中國古算今譚 ——
從傳統數學至西學輸入
至現代課堂數學 V:
現代中學生/明清兩代數學家
初遇上三角函數

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2024.09.21.

中小學數學課堂上傳授的基本知識和技能, 很大部份有數百年以至數千年的歷史. 從古代至十七世紀的東方西方數學典籍當中, 記載了相當多的部份.

回顧及探討從中國古代至十六世紀的傳統數學, 至明末清初西學東漸, 合流會通, 演變為二十世紀以降在華人教育圈中的現代中小學數學課程內容, 不只有其數學意義, 也富有文化意義, 對教與學都有裨益. 從2016年至2023年的四講, 是講者在這方面的嘗試.

中國古算今譚 —— 從傳統數學至西學輸入
至現代課堂數學 2016.05.28.

中國古算今譚 —— 從傳統數學至西學輸入
至現代課堂數學II:現代中學生/明代
徐光啟初遇上綜合幾何 2017.05.27.

中國古算今譚 —— 從傳統數學至西學輸入
至現代課堂數學III:現代中學生/清代
康熙帝初遇上代數方程 2018.05.19.

中國古算今譚 —— 從傳統數學至西學輸入
至現代課堂數學IV:現代中學生/清代
李善蘭初遇上微積分 2023.06.10.

三角學與天文學, 在眾多古代文明中都聯繫在一起, 特別是在古代希臘和印度.
我們今天使用的三角學術語, 許多源自拉丁語名稱, 也來自印度的梵文.
三角學在中世紀的伊斯蘭世界得到進一步發展, 然後傳播到歐洲, 並於十七世紀由耶穌會傳教士將三角學首次傳入中國.

角的概念, 中國自古已有. 但在西學東傳之前, 中國並沒有如古希臘數學那種關於角的闡述, 也就沒有平行線的數學闡述, 因此也沒有如古希臘數學中關於相似三角形的研究. 然而, 那不等於說中國傳統數學沒有測量術的研究. 測量術在中國, 早已有之, 「重差術」更是這方面的出色成就, 等於發展了另外一套替代相似三角形的理論.

本講是系列的第五講, 旨在敘述這段故事, 並旁及三角學教學上的一些問題.

中學數學課程知識增益系列：數學歷史

中學課堂裡三角與幾何的關係

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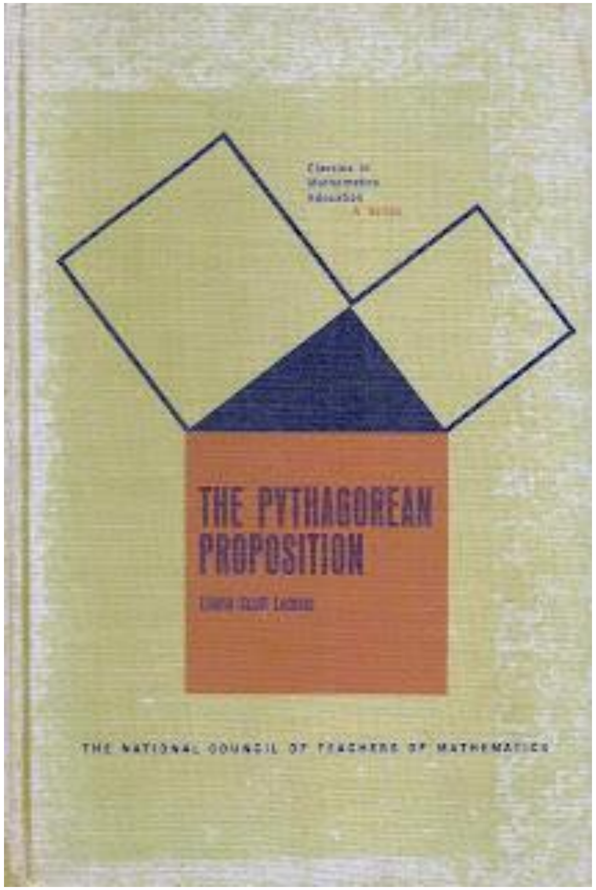
July 28/August 1, 2023.

Two high-school students from New Orleans in Louisiana, USA, Calcea Johnson and Ne'Kiya Jackson, say they have done what math experts claimed was impossible: created a new proof for the 2,000-year-old Pythagorean theorem that is based on trigonometry.

They presented their findings at the AMS Southeastern Sectional Meeting on the Saturday morning of March 18, 2023.

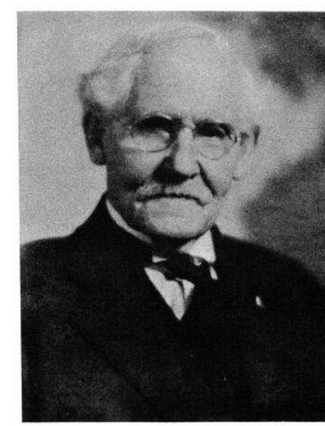
Abstract **In the 2000 years since trigonometry was discovered it's always been assumed that any alleged proof of Pythagoras's Theorem based on trigonometry must be circular. In fact, in the book containing the largest known collection of proofs (*The Pythagorean Proposition* by Elisha Loomis) the author flatly states that "There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem." But that isn't quite true: in our lecture we present a new proof of Pythagoras' Theorem which is based on a fundamental result in trigonometry — the Law of Sines — and we show that the proof is independent of the Pythagorean trig identity $\sin^2 x + \cos^2 x = 1$.**



The Pythagorean Proposition
E. S. Loomis, 1928; 2nd edition 1940; reprinted in 1968, 1972.

The author collected **370 proofs of the Pythagoras' Theorem** in this book. It is an instructive exercise to examine whether all 370 proofs are essentially different, or whether some are closely related to another.

"There are **no trigonometric proofs**, because all the fundamental formulae of trigonometry are themselves based upon the truth of **the Pythagorean Theorem**."



Elisha Scott Loomis (1852-1940)

New Orleans East teens make 'impossible' mathematical discovery unproven for 2,000 years.

Sam Winstrom
11:28 PM EDT March 22, 2023;
updated 5:58 PM EDT March 27,2023.

NEW ORLEANS — It takes a special kind of student to outsmart 2,000 years of mathematicians -- at St. Mary's Academy, they have two. Calcea Johnson and Ne'Kiya Jackson just gave a presentation at the American Mathematical Society's Annual Southeastern Conference. What they were able to do is find a way to prove the Pythagorean Theorem using trigonometry without circular logic --- something mathematicians have been trying to do for nearly 2,000 years. It might not surprise you to hear they were the only high school students in the room.

https://www.wgrz.com/article/news/education/no-east-teens-make-pmathematical-discovery-2000-years-making/289-e89f40cb-85f8-4a55-ae3f-3ec37412d7b8?fbclid=IwAR1VD2xZAyZ4gKcRHbGfl9eU7yYh1TVgE6M4VkbO4vwwum0pzOOHF2EiiEc

New Orleans East teens make ‘impossible’ mathematical discovery unproven for 2,000 years.

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11:28 PM EDT March 22, 2023;
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"It's really an unparalleled feeling, honestly, because there’s just nothing like being able to do something that people don’t think young people can do," Calcea said. "A lot of times you see this stuff, you don't see kids like us doing it."

[...] If you're wondering how two high school seniors figure something like this out, well, it all starts with the teachers who challenged them to do the impossible. "Our slogan is ‘No Excellence Without Hard Labor.' So, they definitely push us," Calcea said.

https://www.wgrz.com/article/news/education/no-east-teens-make-pmathematical-discovery-2000-years-making/289-e89f40cb-85f8-4a55-ae3f-3ec37412d7b8?fbclid=IwAR1VD2xZAyZ4gKcRHbGfl9eU7yYh1TVgE6M4VkbO4vwwum0pzOOHF2EiiEc

New Orleans East teens make 'impossible' mathematical discovery unproven for 2,000 years.

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Really? How should we assess the two schoolgirls' accomplishment in the historical context?

NEW ORLEANS -- It takes a special kind of student to outsmart 2,000 years of mathematicians -- at St. Mary's Academy, they have two. Calcea Johnson and Ne'Kiya Jackson just gave a presentation at the American Mathematical Society's Annual Southeastern Conference.

What they were able to do is find a way to prove the Pythagorean Theorem using trigonometry without circular logic --- something mathematicians have been trying to do for nearly 2,000 years. It might not surprise you to hear they were the only high school students in the room.

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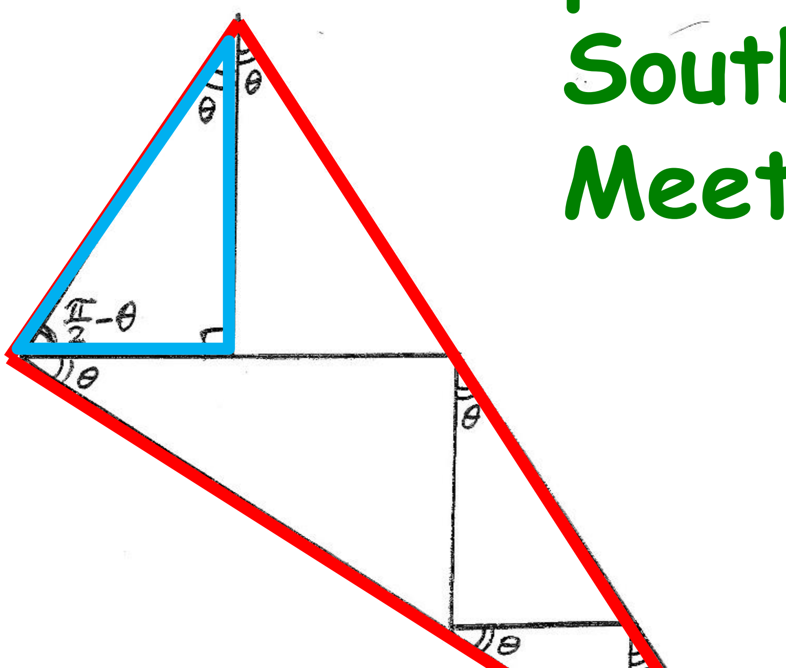
This is the figure drawn by the two schoolgirls in their presentation at the AMS Southeastern Sectional Meeting.



Let us make a good guess and reconstruct the proof they might have given.

This is the given right-angled triangle.

Assume $a < b$ so that $\alpha = a/b < 1$.

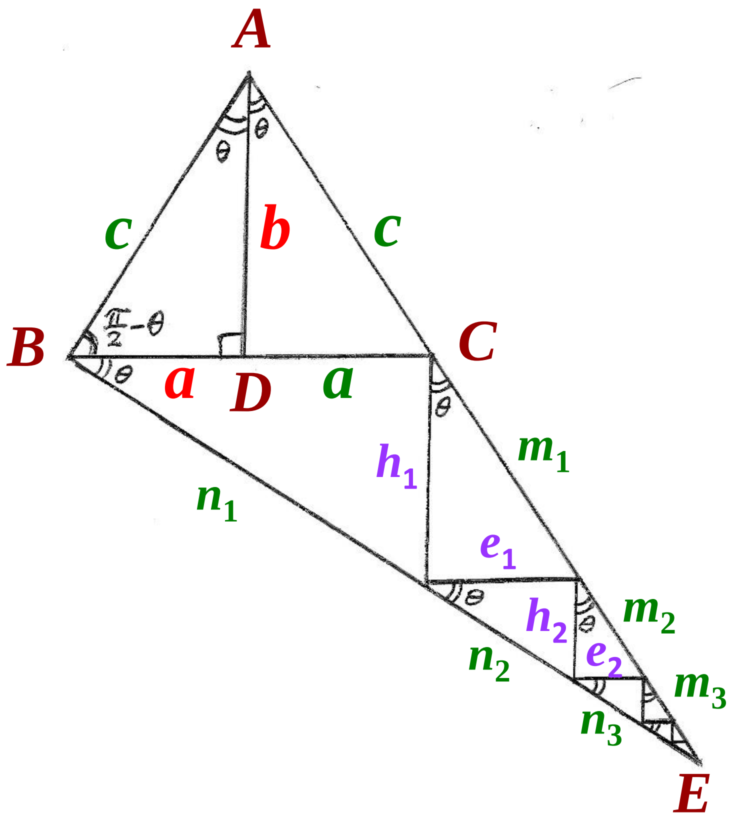
$$a : b : c = h_1 : 2a : n_1 = e_1 : h_1 : m_1$$
$$= h_2 : e_1 : n_2 = e_2 : h_2 : m_2$$
$$= h_3 : e_2 : n_3 = e_3 : h_3 : m_3$$
$$= h_4 : e_3 : n_4 = e_4 : h_4 : m_4$$
$$= \cdots\cdots$$

$h_1 = 2a^2/b = 2\alpha^2 b$, $n_1 = 2ac/b = 2\alpha c$,

$e_1 = 2a^3/b^2 = 2\alpha^3 b$, $m_1 = 2a^2c/b^2 = 2\alpha^2 c$.

$h_k = e_{k-1}^2 / h_{k-1}$, $n_k = e_{k-1} m_{k-1} / h_{k-1}$,

$e_k = h_k^2 / e_{k-1}$, $m_k = n_k e_k / h_k$.

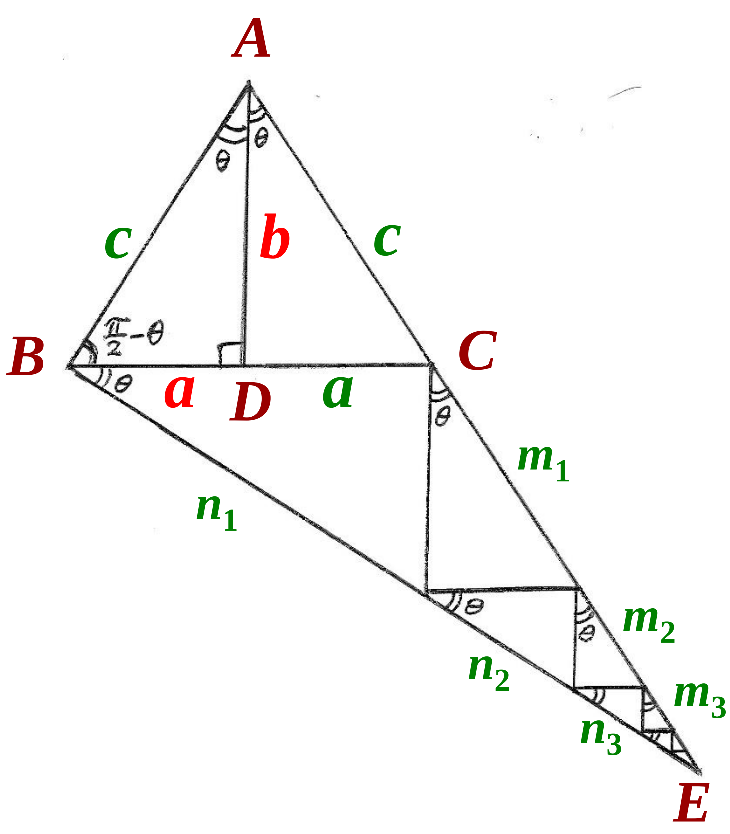


Hence,

$$h_k = 2\alpha^{2k} b,\ n_k = 2\alpha^{2k-1} c,\ e_k = 2\alpha^{2k+1} b,\ m_k = 2\alpha^{2k} c.$$

A probable proof of Pythagoras' Theorem by the two New Orleans schoolgirls (from guess)

$$AE = c + m_1 + m_2 + m_3 + \dots$$
$$= c + 2\alpha^2 c + 2\alpha^4 c + 2\alpha^6 c + \dots$$
$$= c + 2\alpha^2 c\,[1 + \alpha^2 + \alpha^4 + \dots]$$
$$= c + 2\alpha^2 c\,/\,[1 - \alpha^2]$$
$$= c\,(a^2 + b^2)\,/\,(b^2 - a^2)\,.$$
$$BE = n_1 + n_2 + n_3 + \dots$$
$$= 2\alpha c + 2\alpha^3 c + 2\alpha^5 c + \dots$$
$$= 2\alpha c\,[1 + \alpha^2 + \alpha^4 + \dots]$$
$$= 2\alpha c\,/\,[1 - \alpha^2]$$
$$= 2abc\,/\,(b^2 - a^2)\,.$$



A probable proof of Pythagoras' Theorem by the two New Orleans schoolgirls (from guess)

Hence, $\sin 2\theta = BE \,/\, AE$

$= 2ab \,/\, (a^2 + b^2)$.

By the Law of Sines applied to

ΔABC, $\sin 2\theta / 2a = \sin(\pi/2 - \theta)/c$

$= (b/c)\,(1/c)$

$= b/c^2$,

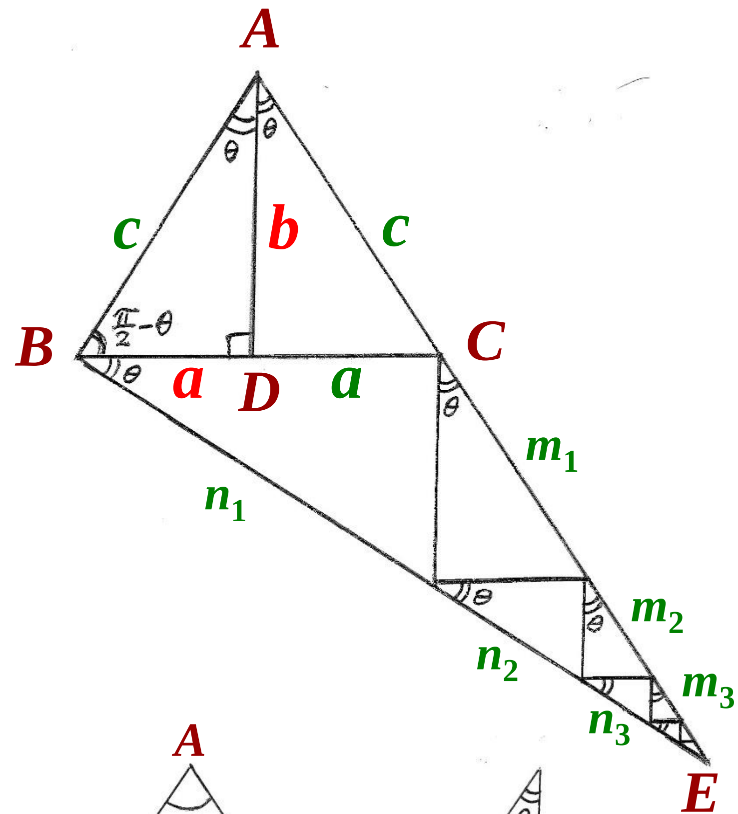
so that $\sin 2\theta = 2ab \,/\, c^2$.

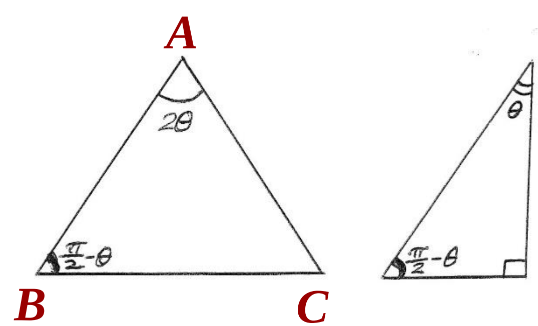
Therefore, $\sin 2\theta = 2ab \,/\, (a^2 + b^2)$

$= 2ab \,/\, c^2$,

from which we conclude that

$$a^2 + b^2 = c^2.$$





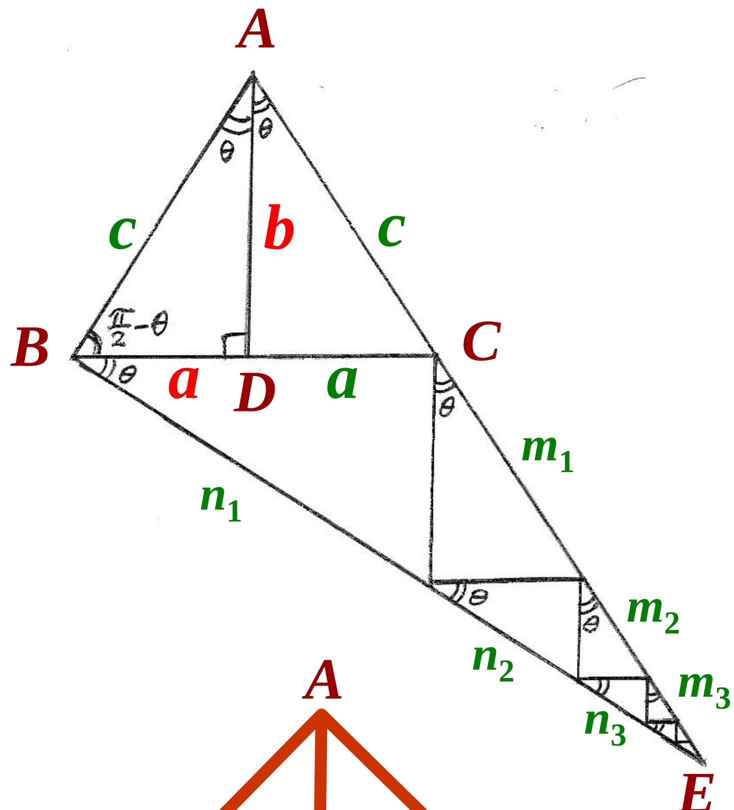
A probable proof of Pythagoras' Theorem by the two New Orleans schoolgirls (from guess)

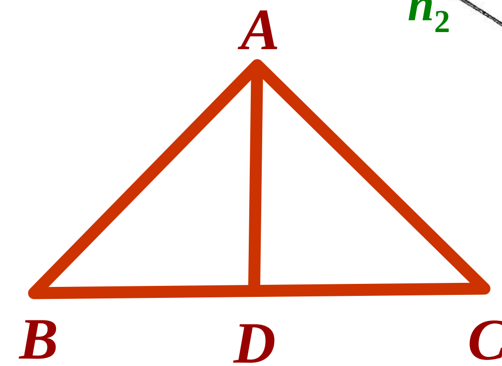
It remains to take care of the case $a = b$. [In this case we cannot form the larger right triangle *ABE*, because *AE* and *BE* will be parallel]. However, this is an easy case to treat.

Since $\Delta\, ABC \sim \Delta\, DBA$,

we have $2a : c = c : a$.

Therefore, $2a^2 = a^2 + b^2 = c^2$.



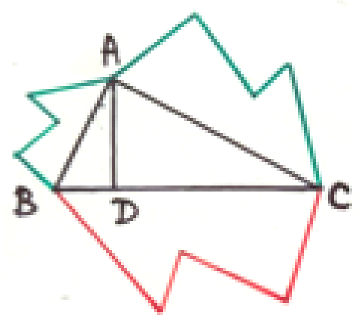


Can we carry out a similar argument in the **general case**?

Yes, we can, and indeed Euclid already carried that out in the proof of **Proposition 31 of *Book VI*** of *his* ***Elements***!

Proposition 31 (*Book VI of Elements*)

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.



Proposition 31 (*Book VI of Elements*)

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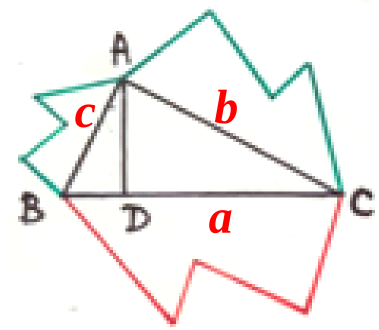
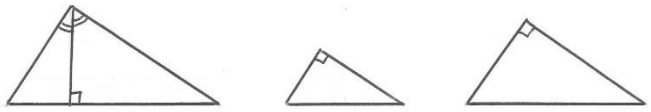


Proof: $\Delta\, ABC \sim \Delta\, DBA$,



$\mathcal{F}(BC)$ $\mathcal{F}(BA)$ $\mathcal{F}(AC)$

so $BC : BA = BA : BD$.

Therefore, $\mathcal{F}(BC) : \mathcal{F}(BA) = BC^2 : BA^2 = BC : BD$.

$\Delta\, ABC \sim \Delta\, DAC$,

so $BC : AC = AC : DC$.

Therefore, $\mathcal{F}(BC) : \mathcal{F}(AC) = BC^2 : AC^2 = BC : DC$.

But $BC = BD + DC$, hence $\mathcal{F}(BC) = \mathcal{F}(BA) + \mathcal{F}(AC)$,

that is, $BC^2 = BA^2 + AC^2$, or $a^2 = c^2 + b^2$. *Q.E.D.*

[$\mathcal{F}$(**) = Figure of a fixed similar shape on ** as one corresponding side.]

It is said that Albert Einstein devised this beautiful proof with the right triangle cut into two smaller similar right triangles when he was a 12-year-old schoolboy.

Albert Einstein
(1879-1955)

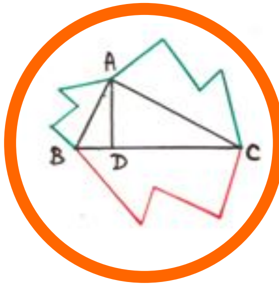
This proof rests on two points:

(1) similar figures

(2) area of similar figures proportional to the square of corresponding sides (Proposition 19 of Book VI of *Elements*)

Proposition 31 (*Book VI of Elements*)

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.



Find three similar figures on each of the three sides such that two of them combine to form the third. These three figures are staring you in the face!

See where the line *ADG* comes from?

Compare with the proof of Proposition 47 in Book I of *Elements*.

$CB : BA = AB : BD$ (because $\Delta CBA \sim \Delta ABD$),

$CB : CA = CA : CD$ (because $\Delta CBA \sim \Delta CAD$).

$\therefore CB^2 : AB^2 = CB : BD$,

$CB^2 : CA^2 = CB : CD$.

Since $CB = BD + CD$, we have

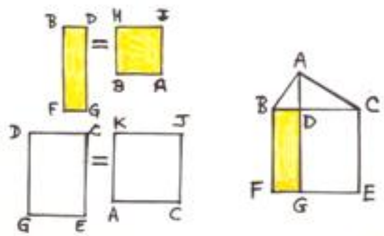
$CB^2 = AB^2 + CA^2$.

$CB : BA = AB : BD$

$\therefore CB \cdot BD = BA^2$

$CB : CA = CA : CD$

$\therefore CB \cdot CD = CA^2$



我怎樣看那兩名路易斯安那州高中女生所做的工作呢？

- 作為兩個高中女生所做的工作, 這是相當出色的成就. 她們的努力和決心、對獨自探索的強烈動力, 當然值得讚揚.

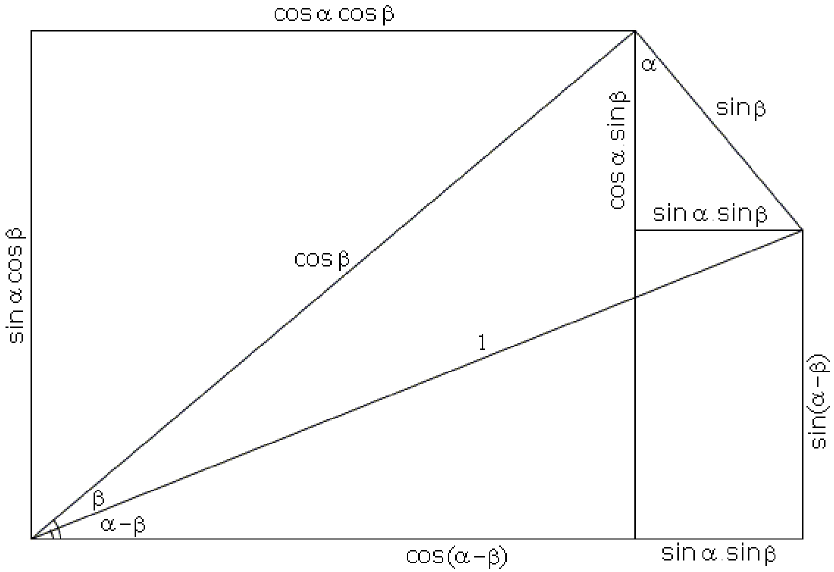
- 她們給畢達哥拉斯定理的證明, 並無涉及循環論證.

- 她們並不是第一個運用三角學去證明畢達哥拉斯定理, 避免了循環論證.

Actually, fourteen years ago a mathematician already published a trigonometric proof of the Pythagorean Theorem without involving the trigonometric identity $(\cos x)^2 + (\sin x)^2 = 1$, which is an equivalent form of the Pythagorean Theorem.

Jason Zimba, On the possibility of trigonometric proofs of the Pythagorean Theorem, *Forum Geometricorum*, Vol. 9 (2009), 1-4.

Abstract. The identity $\cos^2 x + \sin^2 x = 1$ can be derived independently of the Pythagorean theorem, despite common beliefs to the contrary.



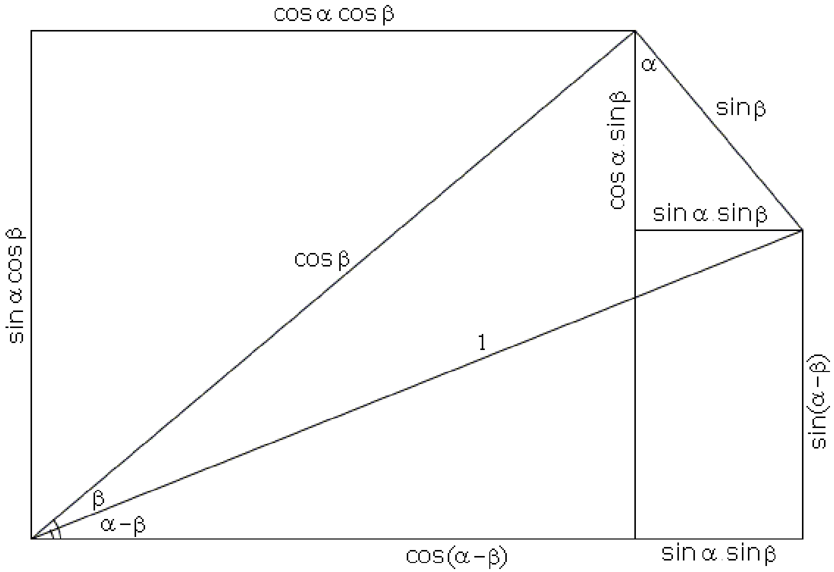
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

To derive these formulae we define the sine and cosine of an acute angle (strictly lying between 0 and π/2) in the standard fashion, thanks to the properties of similar triangles. It is tempting to arrive at the formula

$$(\sin\alpha)^2 + (\cos\alpha)^2 = 1$$

by setting $\alpha = \beta$ in the second formula. But we cannot do so, or else this figure cannot be drawn!



$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

But we can get by with a little trick!

Let $0 < \beta < \alpha < \pi/2$, so $\alpha, \beta, \alpha - \beta$ all strictly lie between 0 and $\pi/2$. Apply the formulae to the right-hand side of $\cos\beta = \cos[\alpha - (\alpha - \beta)]$ twice to arrive at $[(\sin\alpha)^2 + (\cos\alpha)^2]\cos\beta$.

Hence, $\cos\beta = [(\sin\alpha)^2 + (\cos\alpha)^2]\cos\beta$.

By cancelling off $\cos\beta \neq 0$ from each side we obtain $(\sin\alpha)^2 + (\cos\alpha)^2 = 1$.

掩眼法乎?

我怎樣看那兩名路易斯安那州高中女生所做的工作呢？

- 作為兩個高中女生所做的工作, 這是相當出色的成就. 她們的努力和決心、對獨自探索的強烈動力, 當然值得讚揚.

- 她們給畢達哥拉斯定理的證明, 並無涉及循環論證.
- 她們並不是第一個運用三角學去證明畢達哥拉斯定理, 避免了循環論證.
- Elisha Loomis 的名著中關於畢達哥拉斯定理證明的評論, 值得商榷.
- 如果我們深入些考慮「運用三角學去證明」的含義, 便說不上這是兩千多年來數學家都沒有找到的解決方案! 以下, 我們來詳細討論這一點.

Alexandre Borovik, “Decolonisation” of the curricula and some related issues

https://arxiv.org/pdf/2212.13167.pdf (January 12, 2023)

An extended version of this manuscript has been published online on March 23, 2023 in *The Mathematical Intelligencer*, appears in print in Vol. 45, Issue 2 (2023), 144-149.

Abstract. University level mathematics in a number of countries is under pressure to ‘decolonise the curriculum’. [...]

The international mathematical community should defend academic freedom and insist on our right to formulate our curricula and evaluate the history of mathematics and judge mathematicians of the past according to criteria developed within the profession, and ignore any kind of political fads and pressures.

Alexandre Borovik (1956 -)

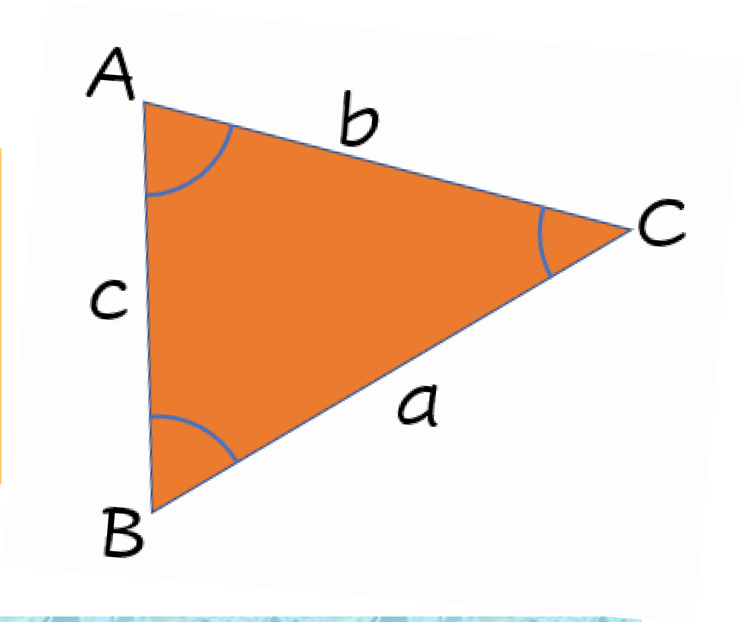
在尋找運用三角學方法證明畢氏(勾股)定理的過程中, 我們不禁要問: 運用三角學方法證明該定理, 究竟是什麼意思?
如果把三角函數僅視為直角三角形某些邊之間的比率的一種“速記語言”, 我們會認為那是運用三角學方法的證明呢, 還是運用幾何學方法的證明呢?

讓我們進一步提出這樣的問題:

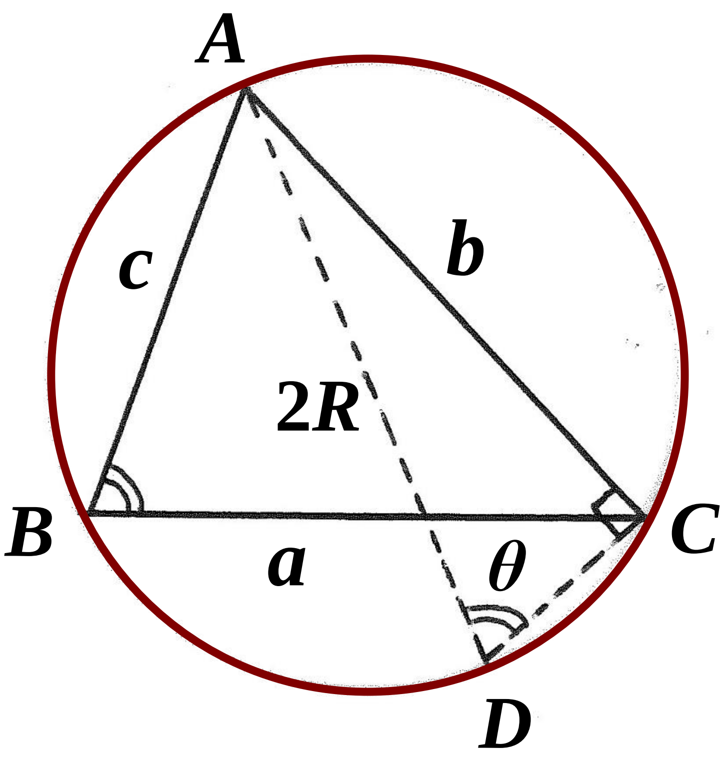
作為學校教授的科目, 幾何學和三角學之間有何聯繫和區別?
三角學在哪一點與幾何學分道揚鑣, 各自繼續走自己的路?

Law of Sines

$a/\sin A = b/\sin B = c/\sin C$



This trigonometric result is basically a geometric property. What is more interesting is to find a geometric meaning of the common value.



AD is the diameter of the circumcircle of ΔABC with radius R.

Hence, angle ACD is a right angle.

$\sin \theta = b / 2R$.

But angle ABC is equal to angle ADC, so $\sin B = \sin\theta = b / 2R$, or $b/\sin B = 2R$.

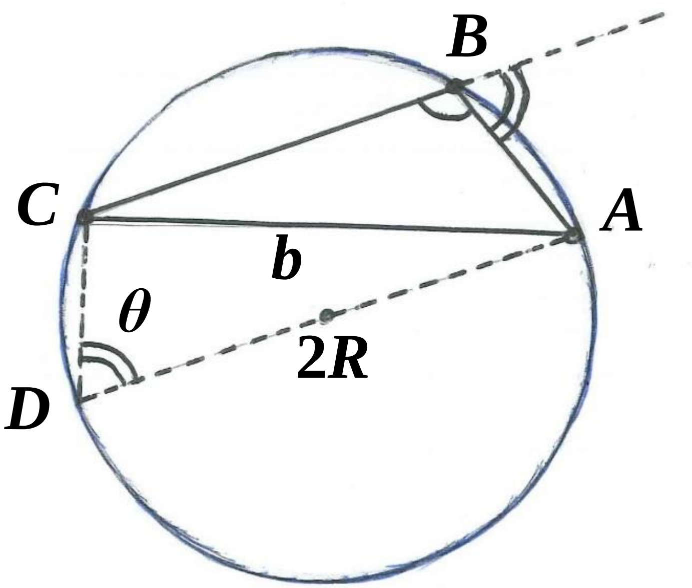
Similarly, $a/\sin A = 2R$, $c/\sin C = 2R$.

Therefore, $a/\sin A = b/\sin B = c/\sin C$, with $2R$ as the common value.

Exercise: Show that this common value is also equal to $abc/2\Delta$, where Δ is the area of ΔABC.

What happens if angle *ABC* is obtuse?

How should we define sin *B* if *B* is obtuse?



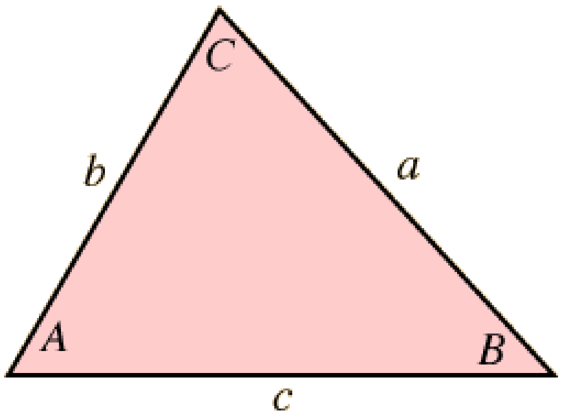
AD is the diameter of the circumcircle of ΔABC with radius *R*.
Hence, angle *ACD* is a right angle.
$\sin \theta = b / 2R$.
But angle *ABC* is supplementary to angle *ADC*, so $\sin \theta = \sin (\pi - B)$.

If it is true that $\sin (\pi - B) = \sin B$, then we can still show that $b/\sin B = 2R$.

This gives a hint of how we should extend the notion of sine to an obtuse angle.

Law of Cosines

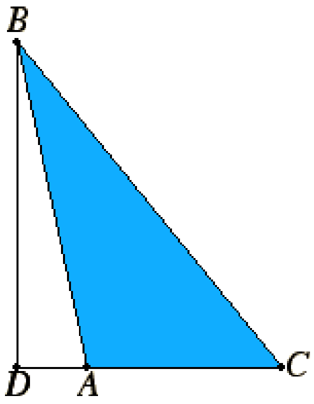
$$c^2 = a^2 + b^2 - 2ab\cos C$$



This trigonometric result is basically a geometric property, which is an extension of Pythagoras' Theorem.

Proposition 12 in Book II of Euclid's *Elements*

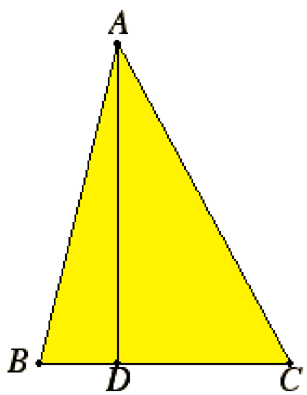
In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.



$$BC^2 = BA^2 + AC^2 + 2 \times AC \times AD$$

Proposition 13 in Book II of Euclid's *Elements*

In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.



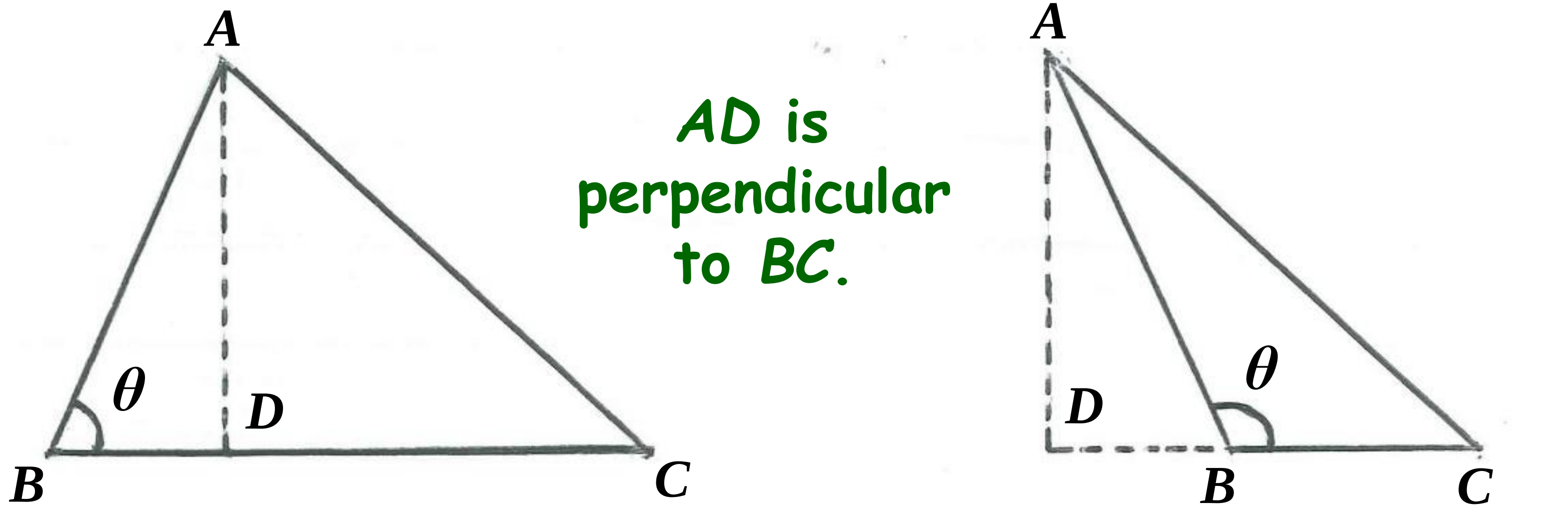
$$BC^2 + BA^2 = AC^2 + 2 \times BC \times BD$$

How can we sum up **both** situations in a **single** formula?

We can put the result as

$$AC^2 = AB^2 + BC^2 - 2 \times AB \times BC \times \cos\theta,$$

where θ is the measure of angle ABC, with the added understanding that $\cos(\pi - \theta) = -\cos\theta$.



$AC^2 = (AB^2 - BD^2) + CD^2$

$= AB^2 + BC^2 - 2 \times BC \times BD$

$AC^2 = AD^2 + (BC + CD)^2$

$= AB^2 + BC^2 + 2 \times BC \times BD$

How can we sum up both situations in a single formula?

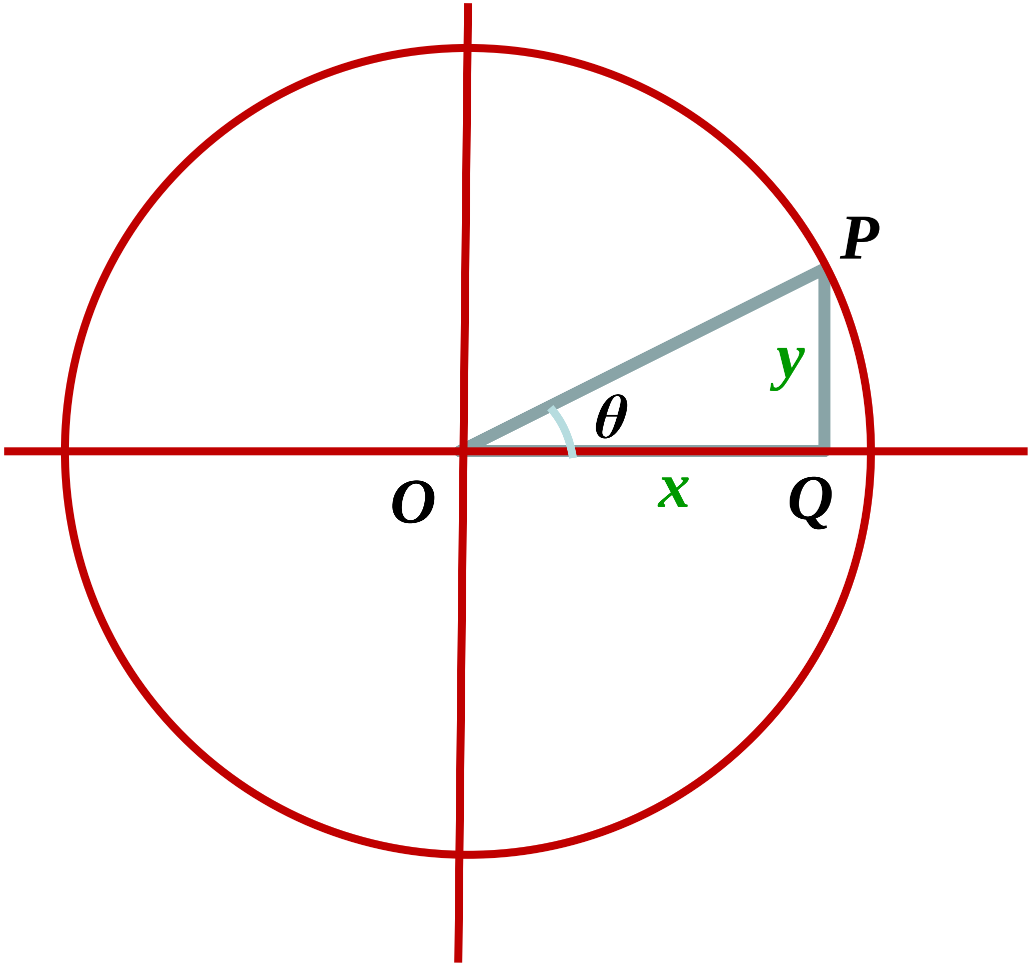
We can put the result as

$$AC^2 = AB^2 + BC^2 - 2 \times AB \times BC \times \cos\theta,$$

where θ is the measure of angle ABC, with the added understanding that $\cos(\pi - \theta) = -\cos\theta$.

This gives a hint of how we should extend the notion of cosine to an obtuse angle.

What about the sine and cosine of an angle larger than 2π ?



$$\sin\theta = \frac{PQ}{OP}$$

$$\cos\theta = \frac{OQ}{OP}$$

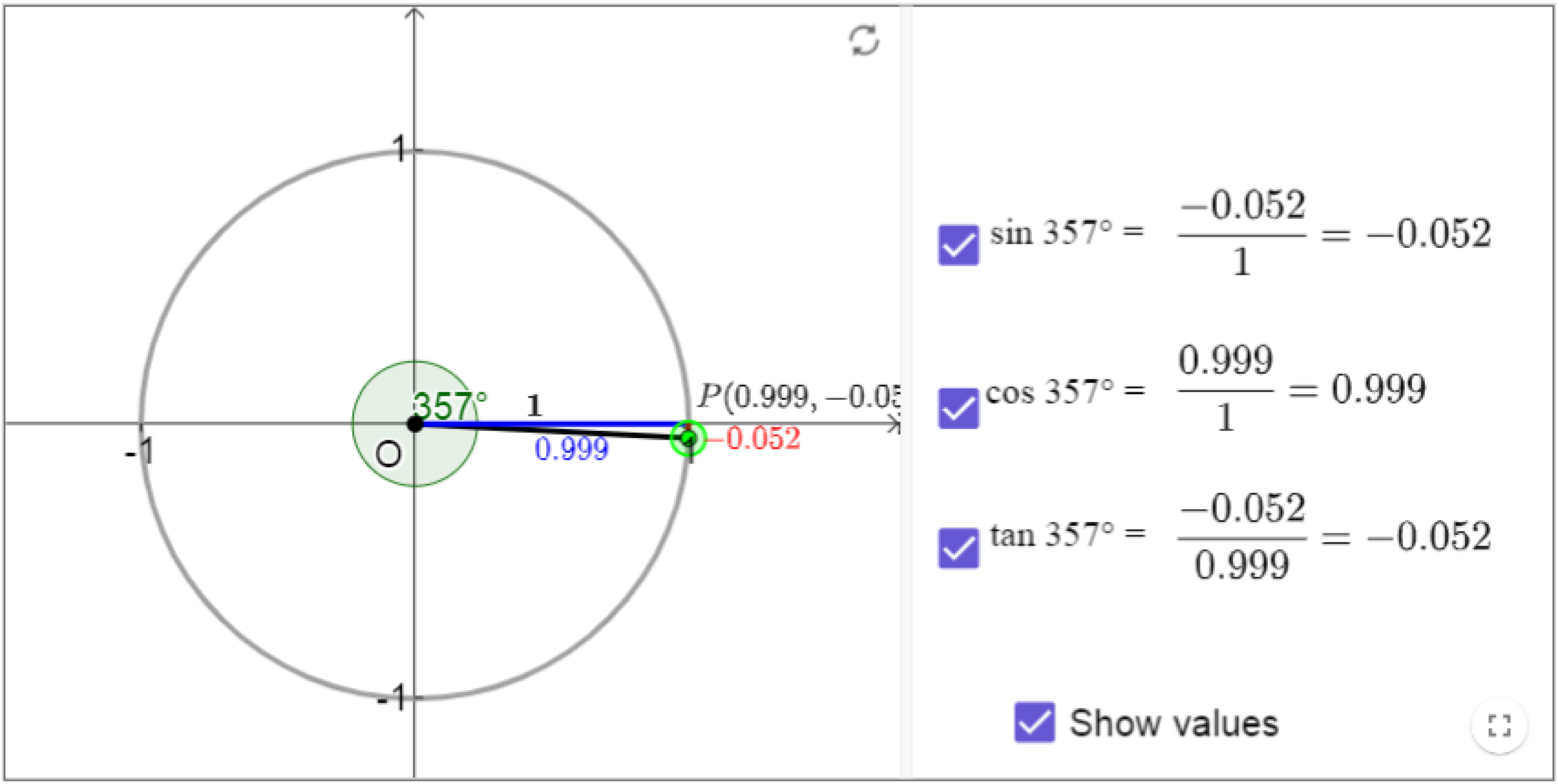
Without loss of generality we may assume $OP = 1$, hence we obtain

$\sin\theta = y$ and $\cos\theta = x$.

How should these notions be generalized to the case $\theta > \pi/2$?

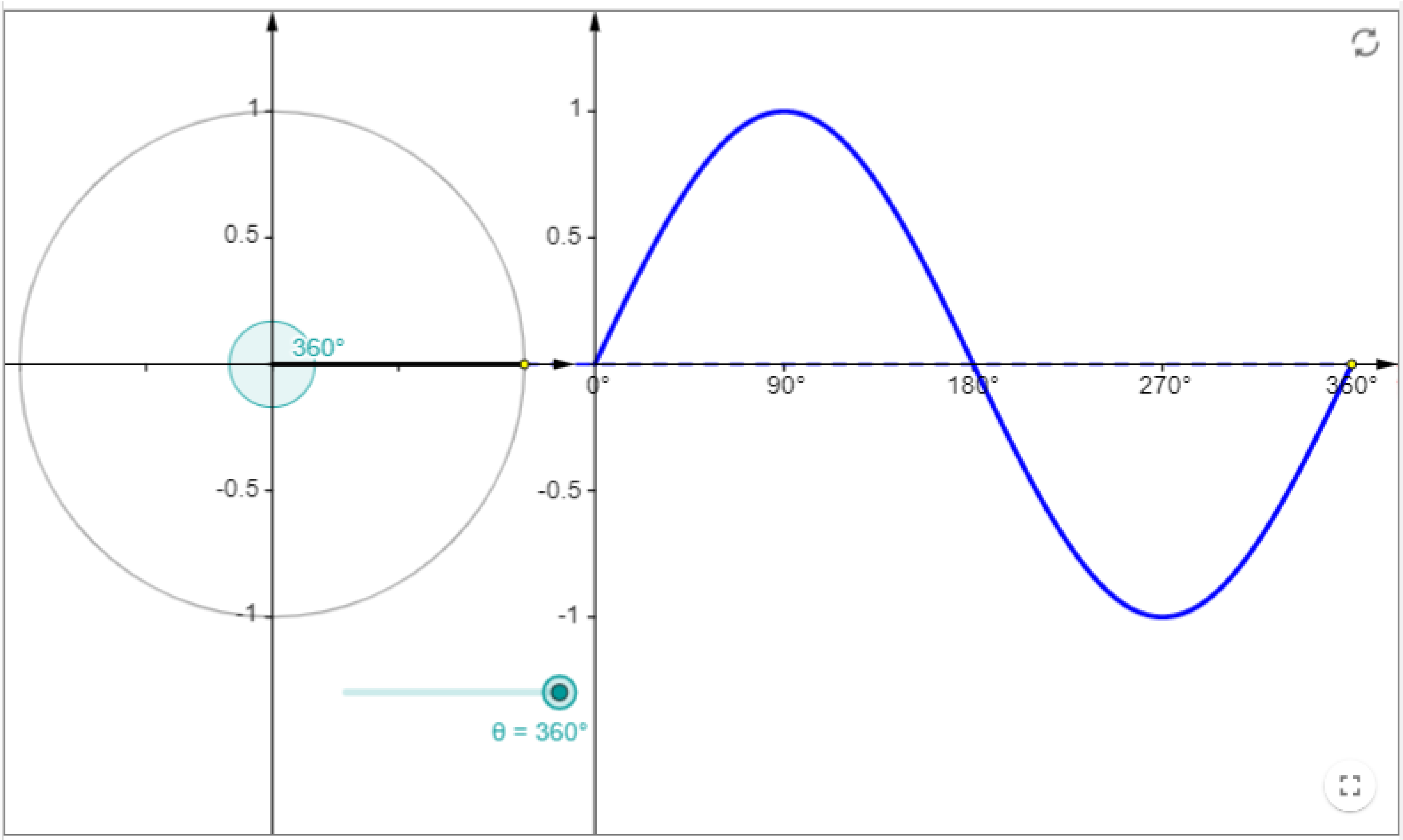
Defining Trigonometric Ratios

https://ggbm.at/Ba2ftYQn



Unit Circle and Sine Graph

https://ggbm.at/S2gMrkbD



Unit Circle and Sine Graph

https://ggbm.at/S2gMrkbD

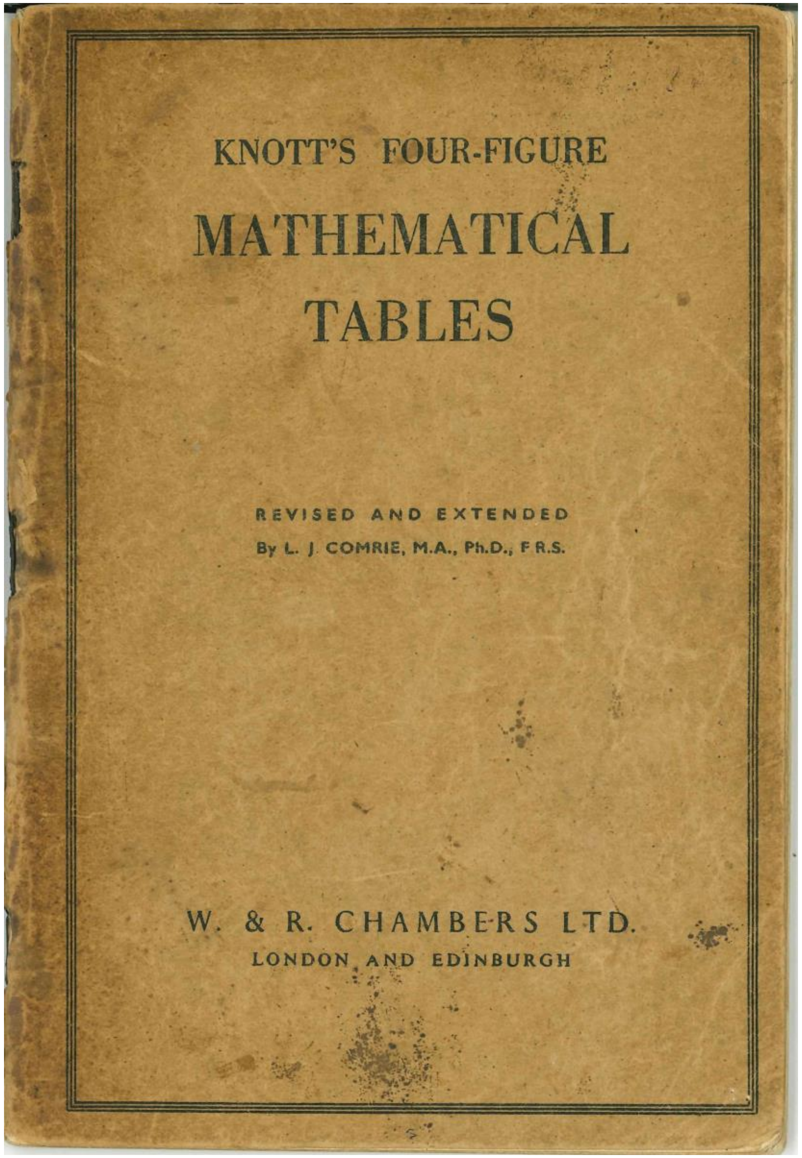
Unit Circle and Cosine Graph

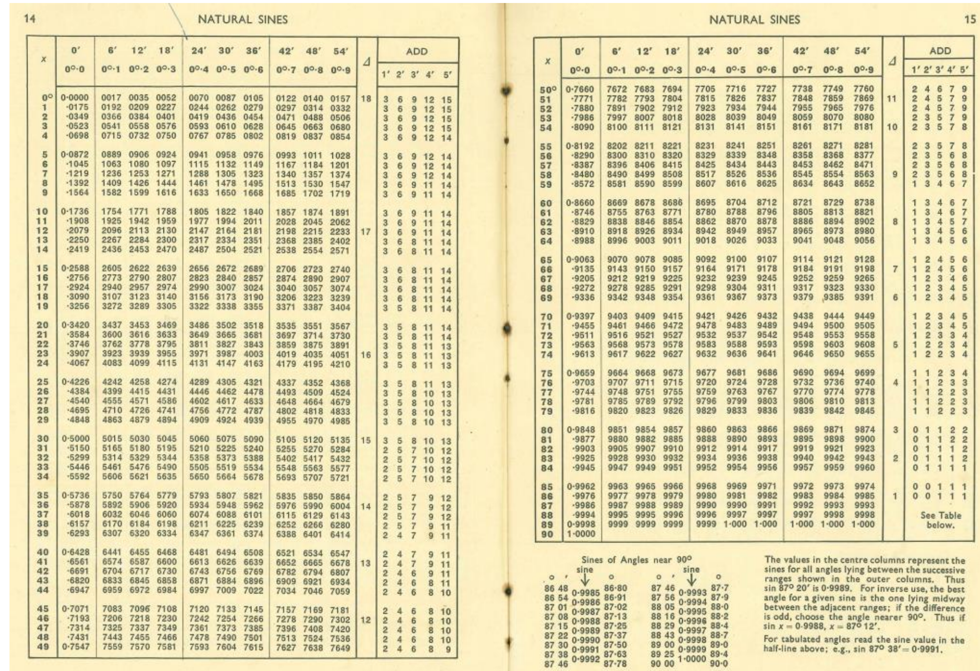
https://ggbm.at/MjFgAfBv

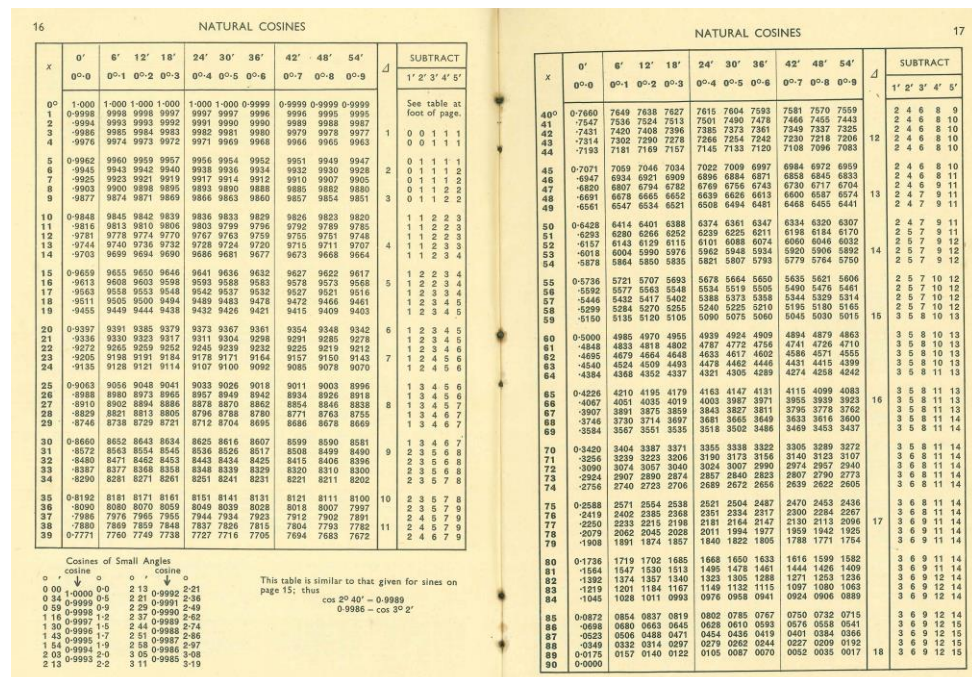
Unit Circle and Tangent Graph

https://ggbm.at/cf6KYJeb

What about the case of an angle with measure larger than 2π ?







Knott's Four-Figure Tables (original edition 1900)

Table of sines

10	0·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891
[illegible]	[illegible]	[illegible]	1942	1959	1977	1994	2011	2028	2045	2062
12	·2079	[illegible]	2113	2130	2147	2164	2181	2198	2215	2233
13	·2250	2267	[illegible]	2300	2317	2334	2351	2368	2385	2402
14	·2419	2436	[illegible]	2470	2487	2504	2521	2538	2554	2571
15	0·2588	2605	2622	[illegible]	2656	2672	2689	2706	2723	2740
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907
17	·2924	2940	2957	2974	[illegible]	3007	3024	3040	3057	3074
18	·3090	3107	3123	3140	[illegible]	3173	3190	3206	3223	[illegible]
19	·3256	3272	3289	3305	3322	[illegible]	3355	3371	[illegible]	3404
20	0·3420	3437	3453	3469	3486	3502	[illegible]	3535	3551	3567
21	·3584	3600	3616	3633	3649	[illegible]	[illegible]	3697	3714	3730
22	·3746	3762	3778	3795	[illegible]	3827	[illegible]	3859	3875	3891
23	·3907	3923	3939	[illegible]	3971	3987	4003	4019	4035	4051
24	·4067	4083	[illegible]	4115	4131	4147	4163	[illegible]	4195	4210
25	0·4226	4242	4258	4274	4289	4305	4321	4337	[illegible]	4368
[illegible]	[illegible]	4399	4415	4431	4446	4462	4478	4493	[illegible]	4524
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	[illegible]
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	[illegible]
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	[illegible]
30	0·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721
35	0·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864
[illegible]	[illegible]	5892	5906	5920	5934	5948	5962	5976	5990	6004

Table of cosines

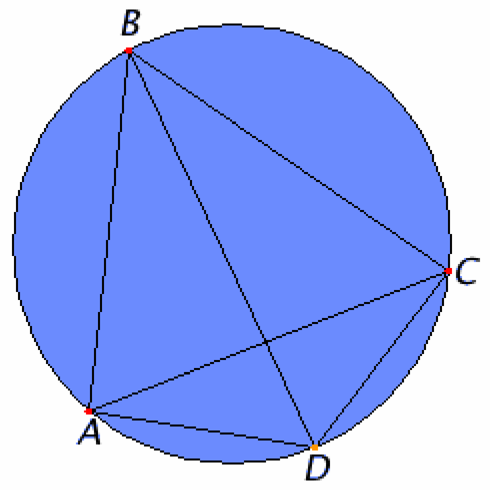
10	0·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820
[illegible]	[illegible]	9813	9810	9806	9803	9799	9796	9792	9789	9785
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664
15	0·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403
20	0·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070
25	0·9063	[illegible]	9048	9041	9033	9026	9018	9011	9003	8996
[illegible]	[illegible]	[illegible]	8973	8965	8957	8949	8942	8934	8926	8918

Cross-multiply and Add

Can you spot that with your naked eyes?

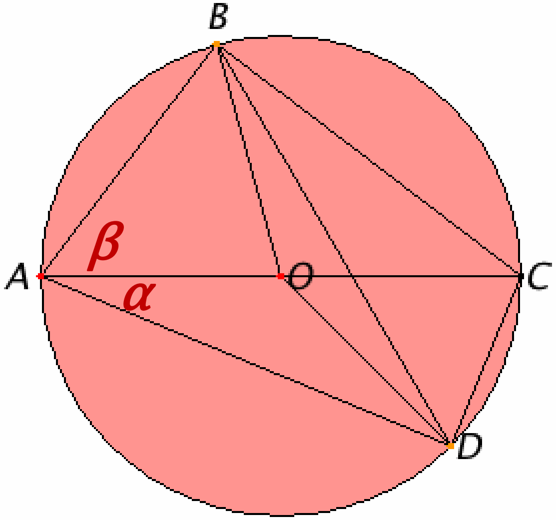
$$0.1736 \times 0.9063 + 0.9848 \times 0.4226 = 0.5736$$

[sin 10 x cos 25 + cos 10 x sin 25 = sin (10 + 25) = sin 35]



Ptolemy's Theorem (2^{nd} Century)

In a cyclic quadrilateral $ABCD$, the product of the diagonals equals the sum of products of opposite sides, that is, $AB.\,CD + BC.\,DA = AC.\,BD$.



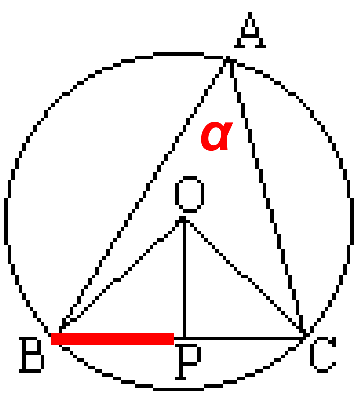
One consequence: **Addition Formula for Sine**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\ .$$

The proof is left as an exercise.

Hint :

sin α = half of the chord subtended by an angle of measure α in a unit circle.
sin α = BP = BC / 2 .

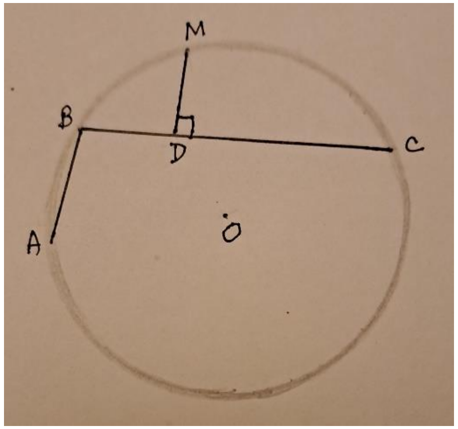


jya-ardha (meaning chord-half in Sanskrit) with shortened form *jiva*
→ *jiba* (in Arabic) with shortened form *jb*, later misinterpreted as *jaib*
→ *sinus* (in Latin) → *sine* (in English)

Broken Chord Theorem

(Archimedes 3rd century B.C.E.)

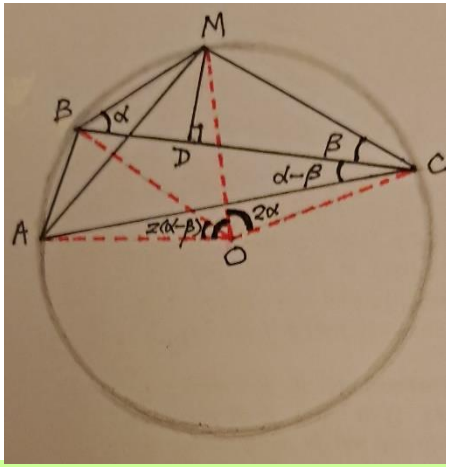
A, B, C are points on a circle with centre O, with $AB \leq BC$. If M is the midpoint of the arc ABC and D is the foot of the perpendicular from M to BC, then D is the midpoint of the broken chord ABC.



One consequence:

Subtraction Formula for Sine

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta .$$



The proof is left as an exercise.
Look at the same hint on half-chord in the case of Ptolemy's Theorem and the Addition Formula for Sine.

$e^{i\theta} = \cos\theta + i\sin\theta$ **Euler's Formula (1740)**

$$\cos(\theta + \varphi) + i\sin(\theta + \varphi) = e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}$$

$$= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$

$$= (\cos\theta\cos\varphi - \sin\theta\sin\varphi)$$

$$+ i(\sin\theta\cos\varphi + \cos\theta\sin\varphi)$$

Hence,

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi,$$

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi.$$

此乃後話 ！

Astronomy

(the oldest science)

→

• • • • •

Trigonometry

• • • • •

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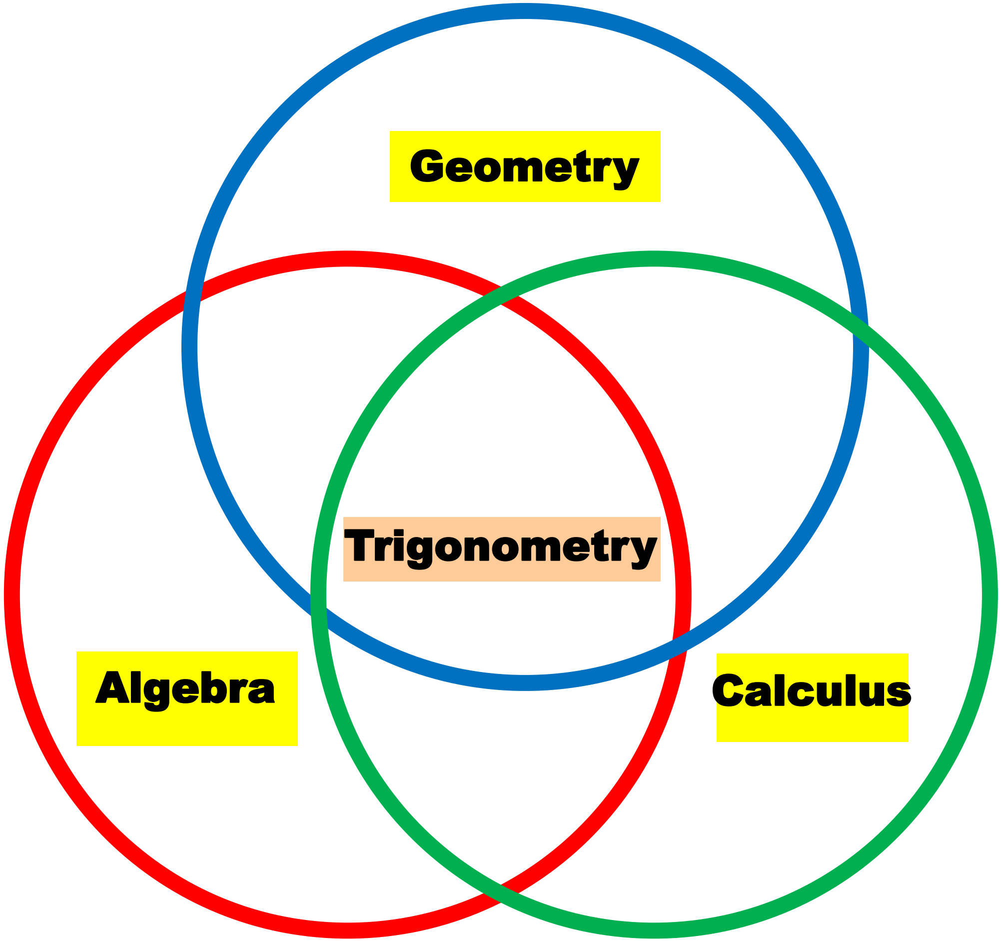
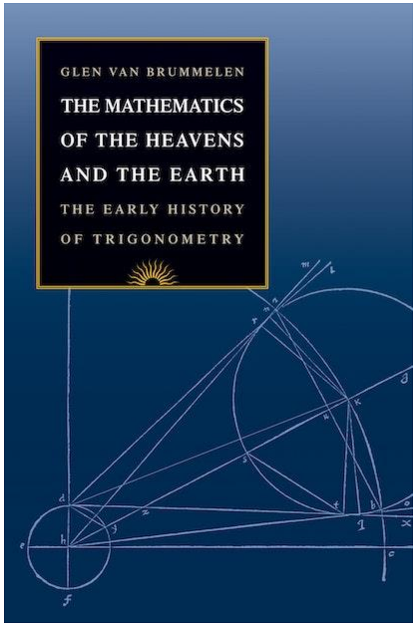
teaching sequence of trigonometry

≠ historical development

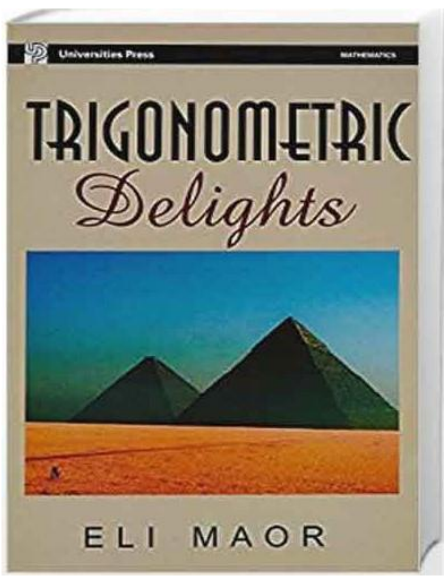
BUT there are many touching points and intersections, making an integration of historical material in the learning and teaching of the subject possible and valuable.



A more appropriate analogy is a composition of various themes (with variations) like in a symphony.



Glen van Brummelen, ***The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*** **(2009)**



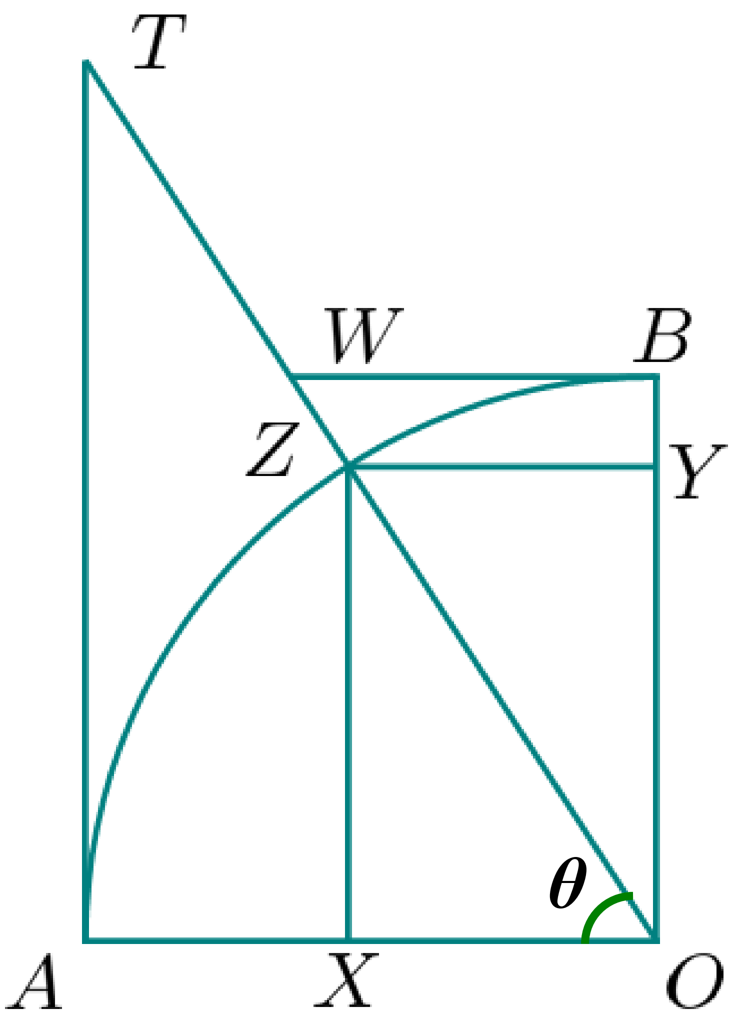
Eli Maor, ***Trigonometric Delights*** **(1989)**

- 在大多數古代文明中, 三角學都與天文學聯繫在一起, 特別是在古代希臘和印度. 我們今天使用的三角學術語, 許多源自拉丁語名稱, 而這些拉丁語名稱又來自印度的梵文.

- 三角學在中世紀的伊斯蘭世界得到進一步發展, 然後傳播到歐洲. 歐洲第一本關於三角學的書本, 大抵是 Bartholomaeus Pitiscus 於 1595年編寫的 *Trigonometria : sive de solutione triangulorum tractatus brevis et perspicuus* .

- 從此, 三角學成為歐洲數學的一部分, 並於17世紀由瑞士耶穌會士鄧玉函 [Jean Terrenz] 撰寫《大測》一書, 作為《崇禎曆書》的一部分, 將三角學首次傳入中國.《崇禎曆書》是由明朝大臣徐光啟於1629 年設立的曆局編成的鉅著, 共137卷.

割圓八線圖



OXAZBYO 是單位圓的四分一.
Δ *OAT* 是直角三角形.
BW 和 *AT* 是圓的切線,
BW 與 *YZ* 平行, *AT* 與 *XZ* 平行

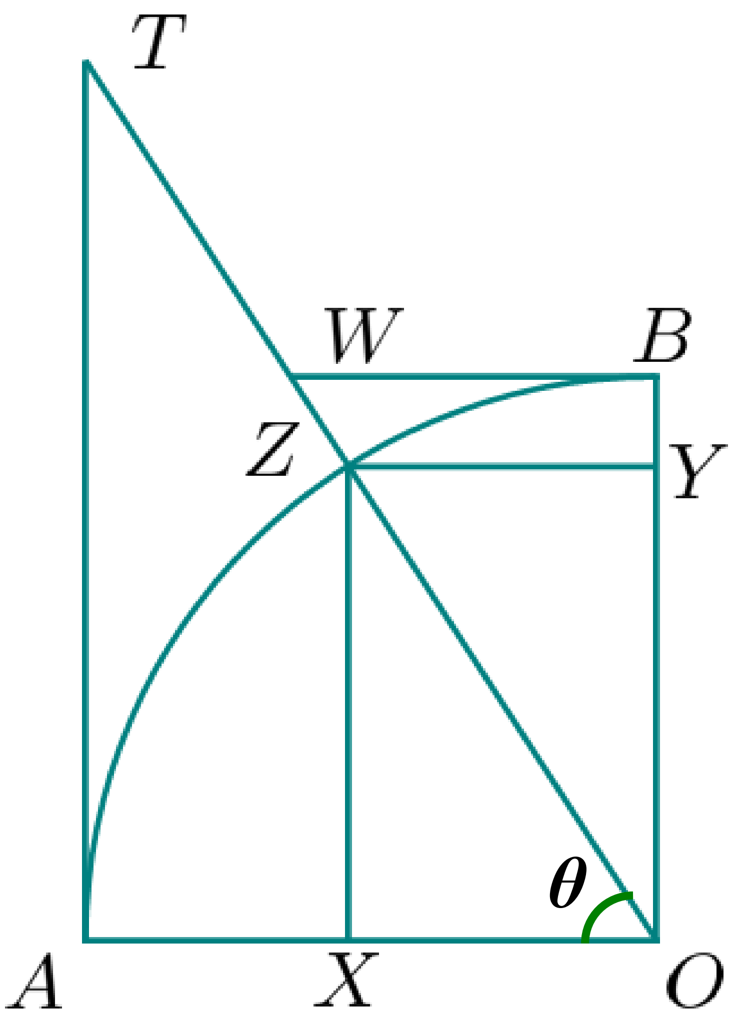
XZ = 正弦 (sine)	*YZ* = 餘弦 (cosine)
AT = 正切 (tangent)	*BW* = 餘切 (cotangent)
OT = 正割 (secant)	*OW* = 餘割 (cosecant)
XA = 正矢 (versine)	*YB* = 餘矢 (coversine)

$\text{versin}\ \theta = 1 - \cos\theta$

$\text{coversin}\ \theta = 1 - \sin\theta$

Note: $\text{vercos}\ \theta = 1 + \cos\theta \neq \text{coversin}\ \theta$

割圓八線圖



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Δ *OAT* 是直角三角形.
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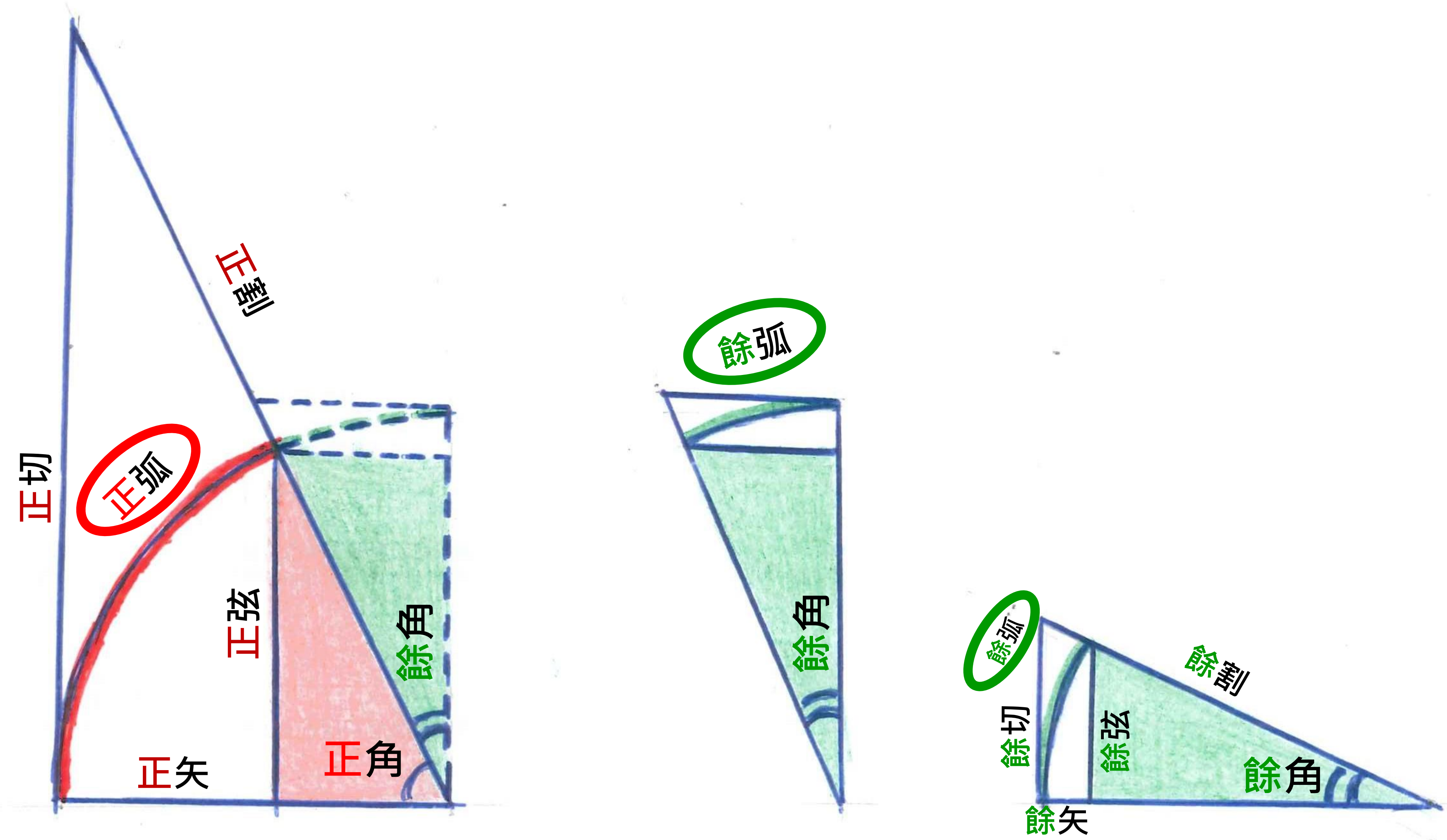
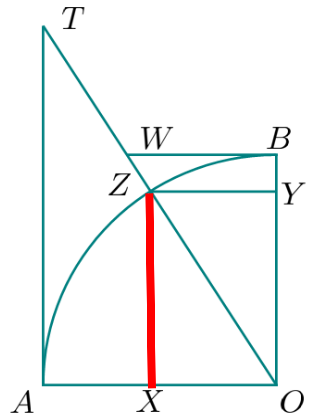
BW 與 *YZ* 平行, *AT* 與 *XZ* 平行

AZ = 正弧

BZ = 餘弧


正割
正切
正弧
正弦
餘角
正矢
正角
餘弧
餘角
餘弧
餘割
餘切
餘弦
餘角
餘矢

清代數學家梅文鼎以明末傳教士編譯的《大測》,《測量全義》等書為基礎, 對有關三角形的算法作了系統的整理, 寫成《平三角舉要》.



正弦為八線之主
[Sine is the Master of the Eight Lines].

梅文鼎
(1633-1721)

西法用三角, 猶古法之用句股也. 但三角有鈍角, 而句股無之, 論者遂謂句股之數有所窮, 殊不知銳角形須分為兩句股, 鈍角形須補成句股, [...], 然則句股雖不能備三角之形, 而能兼三角之理, 三角不能出句股之外, 而能盡句股之用, 一而二, 二而一者也.

是耶？ 非耶？

梅文鼎 《平三角舉要》(十七世纪)

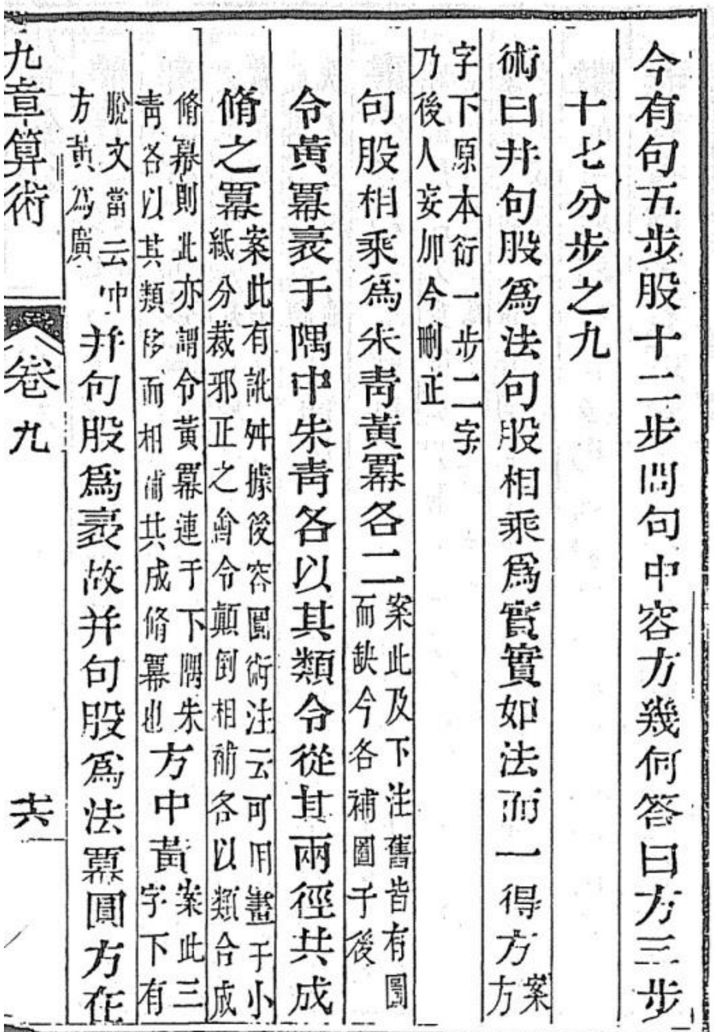
角的概念, 中國自古已有. 但在西學東傳之前, 中國並沒有如古希臘數學關於角的那種闡述, 也沒有平行線的數學闡述;因此, 也就沒有如古希臘數學中關於相似三角形的研究.

然而, 那不等於說中國傳統數學沒有測量術的研究. 測量術在中國, 早已有之, 「重差術」更是這方面的出色成就.

中國傳統數學運用面積來計算同類的問題, 基於「勾(股)中容橫, 股(勾)中容直, 二積皆同」這個原理,中國數學家發展了另外一套替代相似三角形的理論.

《九章算術》第九章第十五題

（成書於公元前一世紀至公元一世紀之間）



今有句五步股十二步問句中容方幾何答曰方三步

十七分步之九

術曰并句股為法句股相乘為實實如法而一得方 案方字下原本衍一步二字乃後人妄加今刪正

句股相乘為朱青黃冪各二 案此及下注舊皆有圖而缺今各補圖于後

令黃冪袤于隅中朱青各以其類令從其兩徑共成

脩之冪 案此有訛舛據後容圓術注云可用畫于小紙分裁邪正之會令顛倒相補各以類合成脩冪則此亦謂令黃冪連于下隅朱青各以其類移而相補共成脩冪也 方中黃 案此三字下有脫文當云中方黃為廣 并句股為袤故并句股為法冪圓方在

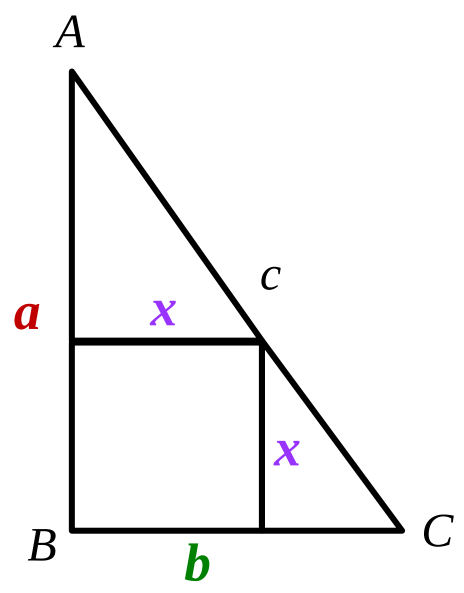
九章算術 卷九 共

術曰：並**勾股**為法.

勾股相乘為實.

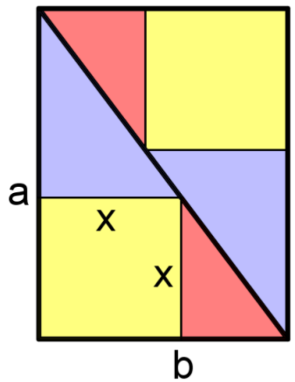
實如法而一得方.

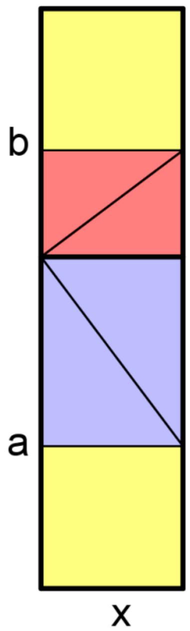
$$x = ab/(a + b)$$



劉徽注《九章算術》[公元三世紀中葉]

方法 1 (出入相補原理)





Area = *ab*   Area = (*a* + *b*) *x*

ab* = (*a* + *b*) *x

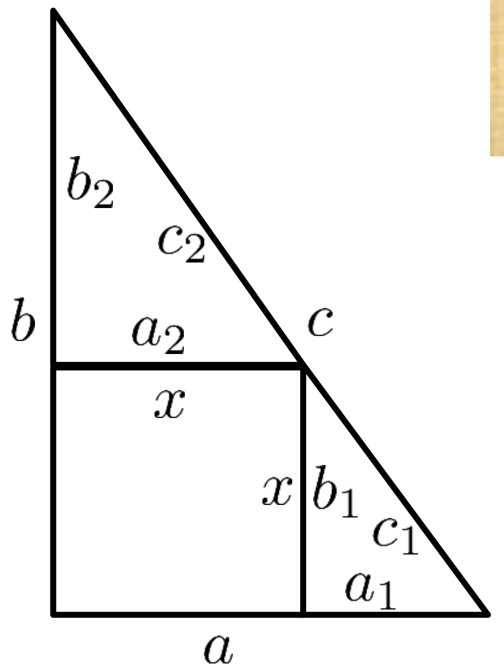
$$x = \frac{ab}{a+b}$$

http://ggbtu.be/m2812253

劉徽注《九章算術》[公元三世紀中葉]

方法 2 (比率)

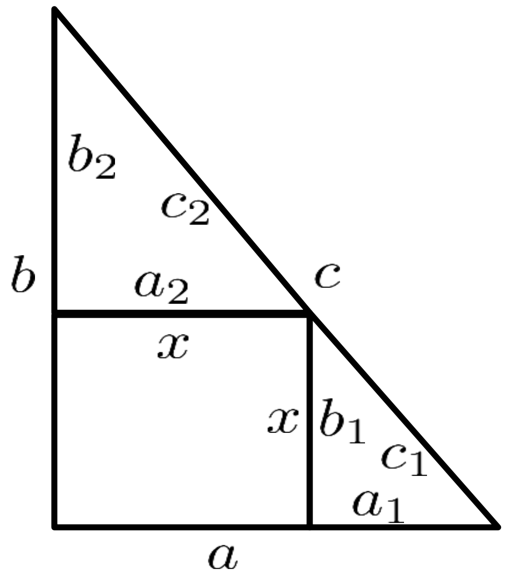
方在勾中，則方之兩廉各自成小勾股，而其相與之勢不失本率也。



$$a : b : c = a_1 : b_1 : c_1$$

$$\therefore\ (a+b) : b = (a_1 + b_1) : b_1 = a : x$$

即是 $x = ab / (a+b)$

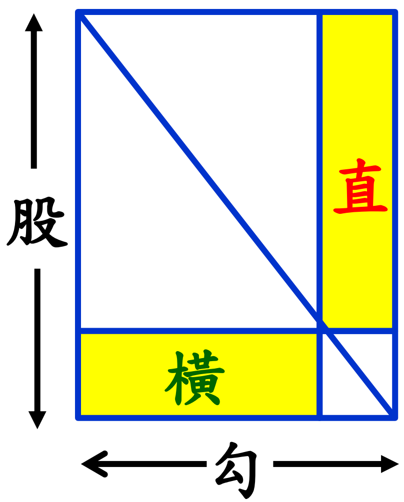


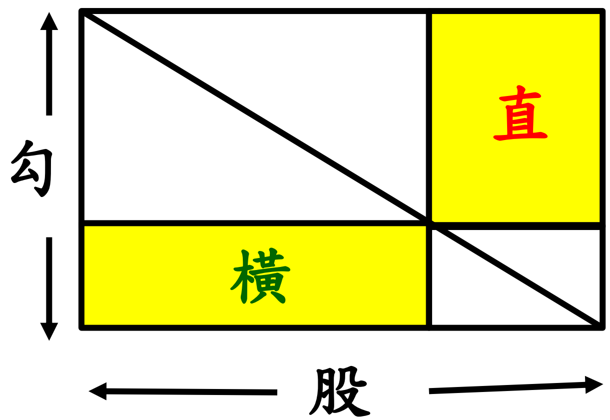
$$a : b : c$$
$$= a_1 : b_1 : c_1$$
$$= a_2 : b_2 : c_2$$

今天, 我們運用相似三角形的知識, 很容易得出這種關係. 但是, 幾千年前的中國數學家**如何處理**這個問題呢?

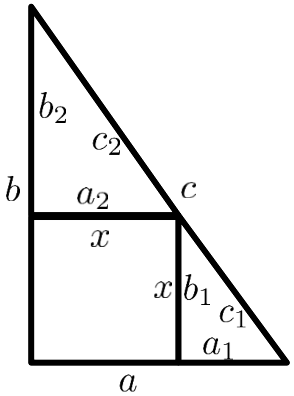
勾(股)中容横。股(勾)中容直。二積皆同。

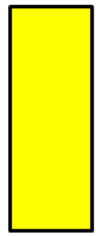
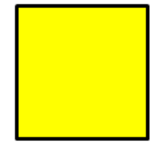
楊輝
《續古摘奇算法(卷下)》
(1275)



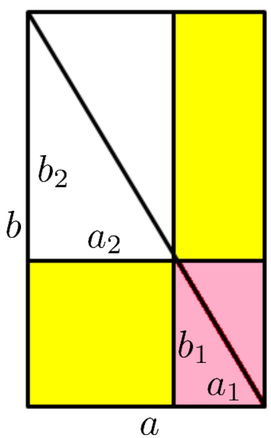


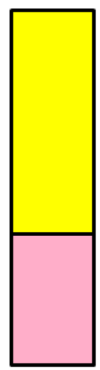
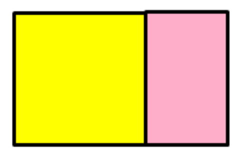
Compare with Proposition 43 of Book I of Euclid's *Elements* (ca. 300 B.C.E.), which applies more generally to a parallelogram.



 $= a_1 b_2,$  $= a_2 b_1.$

Hence, $a_1 b_2 = a_2 b_1$, or $a_1 : a_2 = b_1 : b_2$.



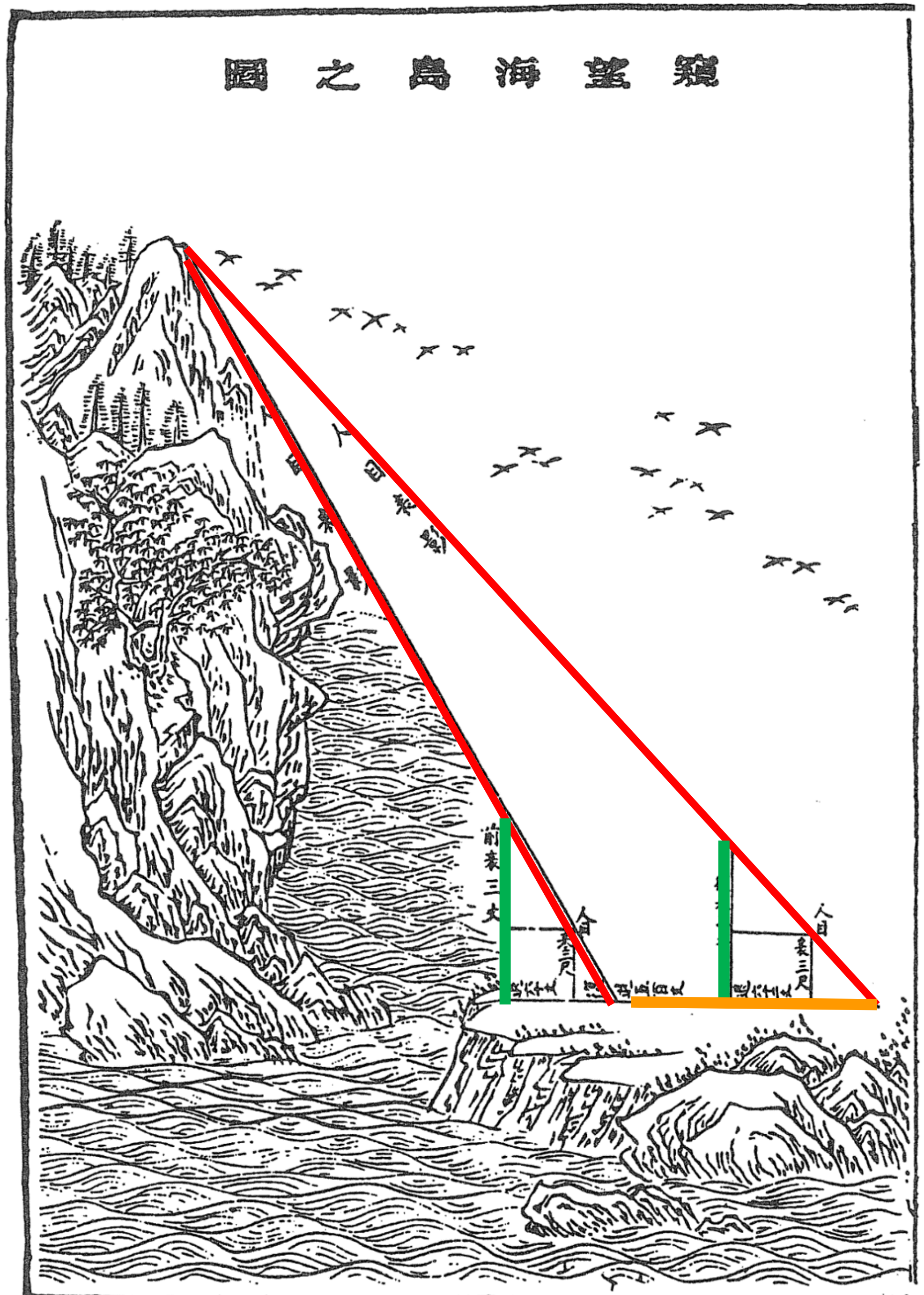
 $= a_1 b,$  $= ab_1.$

Hence, $a_1 b = ab_1$, or $a : a_1 = b : b_1$.

Since $c^2 = a^2 + b^2, c_1^2 = a_1^2 + b_1^2, c_2^2 = a_2^2 + b_2^2$,

we have $a : a_1 : a_2 = b : b_1 : b_2 = c : c_1 : c_2$,

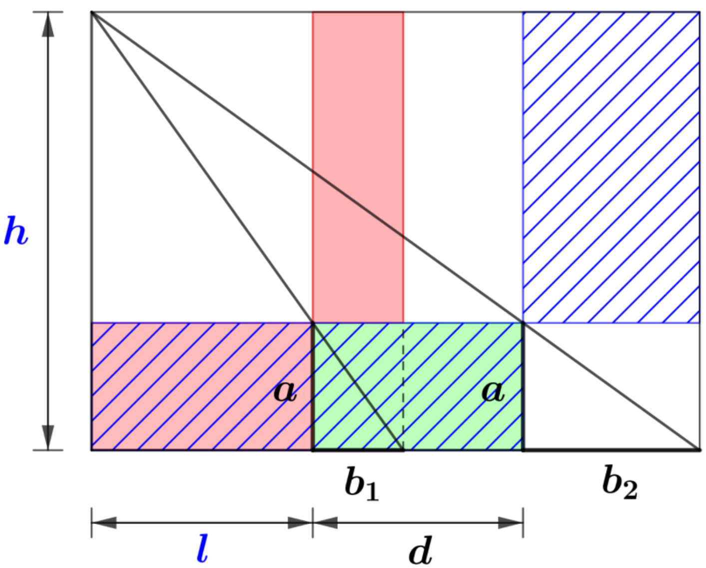
or $a : b : c = a_1 : b_1 : c_1 = a_2 : b_2 : c_2$.

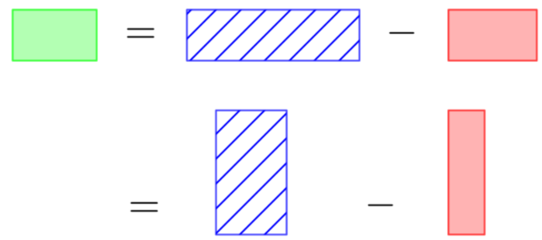


Method of Double-Difference [重差術] of LIU Hui [劉徽]

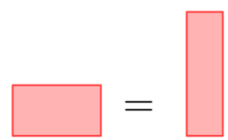
in ***Haidao Suanjing***
[海島算經 Sea Island Mathematical Manual]
(3rd century) as illustrated
in *Gujin Tushu Jicheng*
[古今圖書集成 Complete Collection of Pictures and Writings of Ancient and Modern Times] (1726)

Given a, d, b_1 and b_2, how can we express h and l in terms of a, d, b_1 and b_2 ?





$$\therefore\ ad = (h-a)b_2 - (h-a)b_1$$
$$= (h-a)(b_2-b_1)$$
$$h = \frac{ad}{b_2-b_1} + a$$



$$\therefore\ la = b_1(h-a) = \frac{b_1 ad}{b_2-b_1}$$
$$l = \frac{b_1 d}{b_2-b_1}$$

Explanation by YANG Hui [楊輝] on the Method of Double-Difference of LIU Hui [劉徽] (1275)

http://ggbtu.be/m2812113

Aryabhata I

阿耶波多

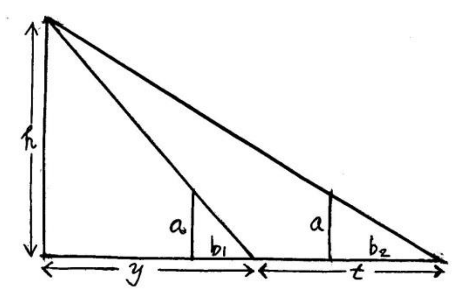
(c.476-550)

Aryabhatiya

《阿耶波多曆數表》

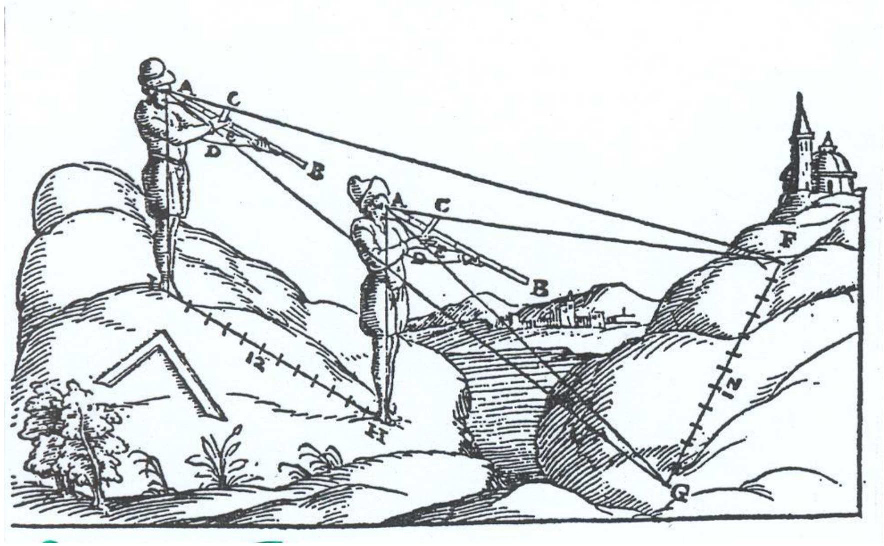
Book II, Stanza 16

The distance between the ends of the two shadows multiplied by the length of the first shadow and divided by the difference in length of the two shadows gives the *kotī* . The *kotī* multiplied by the length of the gnomon and divided by the length of the (first) shadow gives the length of the *bhujā* .



$$y = \frac{tb_1}{b_2 - b_1}$$

$$h = \frac{ya}{b_1} \quad \left(= \frac{ta}{b_2 - b_1} \right)$$



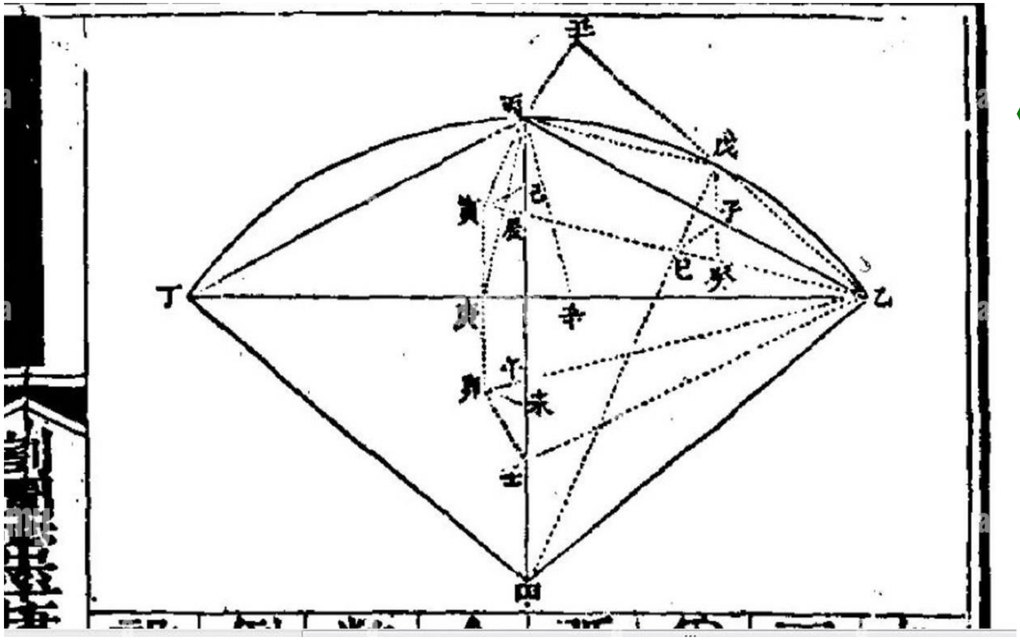
Orence Fine, *De re & praxi geometrica* (1556)

A Jacob's staff, from John Sellers' *Practical Navigation* (1672)

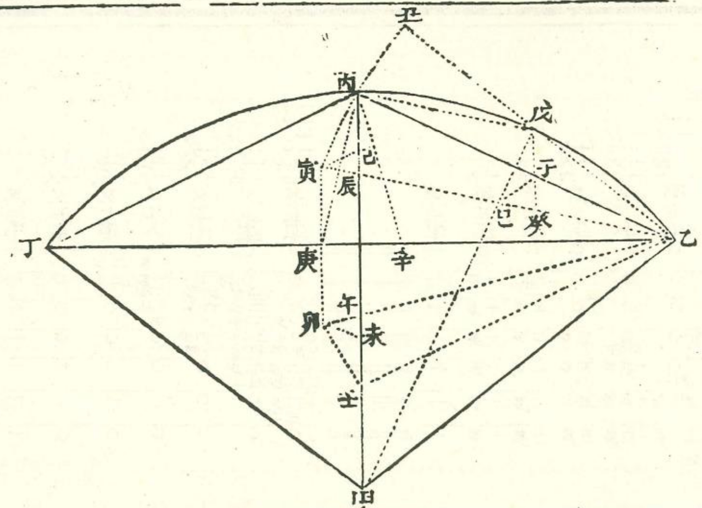
John Sellers, *Practical Navigation* (1672)

The invention of the cross-staff (or Jacob's staff) has been credited to Levi ben Gerson (1288-1344).

明安圖[Ming Antu] (~ 1692-1763)



明安圖
《割圓密率捷法》
1839 (~1730-1763)



如圖甲爲圜心甲乙類爲半徑乙丙丁爲圜周之一弧乙丙弧丙丁弧俱爲二分弧之一爲一分弧乙丁爲全弧通弦乙丙丙丁俱爲一分弧通弦今命甲乙類爲連比例第一率乙丙類爲連比例第二率求乙丁當連比例之率數是三線本非一連比例欲設法以取之也試任平分一分弧乙

又三率	一六 又五率	一六 一六 又七率	一六 一六 一六 又九率	一六 一六 一六 一六 又十一率	一六 一六 一六 一六 一六 又十三率	一六 一六 一六 一六 一六 一六 又十五率
一	一少 一二	〇 二少	〇 一多	〇	〇	〇
一	〇	二少 二三	一多 六多少 六五少 一五	〇 六多少 六〇多少	〇 二少 三少	〇
一	〇	〇	〇	二〇少 二四少 一四	二〇多 二八多 七〇少	〇 二〇少 三〇少 一四〇多
一	〇	〇	〇	〇	四二少 四二	一二〇多 二五二少
一	〇	〇	〇	〇	〇	一三二少 一三二
一	〇	〇	〇	〇	〇	〇

第三率	一六 第五率	一六 一六 第七率	一六 一六 一六 第九率	一六 一六 一六 一六 第十一率	一六 一六 一六 一六 一六 第十三率	一六 一六 一六 一六 一六 一六 第十五率
一	〇 一	〇	〇	〇	〇	〇
一	一多	〇 二	〇	〇	〇	〇
一	一多	二多	〇 五	〇	〇	〇
一	一多	二多	五多	〇 一四	〇	〇
一	一多	二多	五多	一四多	〇 四二	〇
一	一多	二多	五多	一四多	四二多	〇 一三二
一	一多	二多	五多	一四多	四二多	一三二多

率之一少五次十六分十三率之五多六次十六分十五率之十與四次十六分十一率之一二得數爲同母相等次另書十一率同母相等數于上三率相等數于下乘除得五次十六分十三率之一少六次十六分十五率之六與五次十六分十三率之一二得數爲同母相等次另書十三

明安圖•《割圜密率捷法》 1839 (~1730-1763)

割圜密率捷法序

欽天監監正明靜菴先生自童年親受數學於

聖祖仁皇帝至老不倦病革時以遺稿一帙囑其季子景臻命

際新續而成之日此割圜密率捷法也內圜徑求周弧背求弦

求矢三法本泰西杜氏德美所著實古今所未有也亟欲公諸

同志惜僅有其法而未詳其義恐人有金針不度之疑予積解

有年未能卒業汝與同學者務續而成之則予志也　先生沒

際新尋繹推究質以平日所聞面授之言遇有疑義則與　先

Three methods transmitted by the French Jesuit Pierre Jartoux (1669-1720) in 1701 —— infinite series for π, for sine and for versine

明安圖

《割圓密率捷法》1839

陳際新 序

昔人云:「鴛鴦繡出從君看, 不把金針度與人」, 吾輩言幾何之學, 政與此異. 因反其語曰:「金針度去從君用，未把鴛鴦繡與人」. 若此書者, 又非止金針度與而已, 直是教人開丱冶鐵, 抽線造計; 又是教人植桑飼蠶, 湅絲染縷. 有能此者, 其繡出鴛鴦, 直是等閑細事. 然則何故不與繡出鴛鴦? 曰: 能造金針者能繡鴛鴦, 方便得鴛鴦者誰肯造金針? 又恐不解造金針者, 菟絲棘刺, 聊且作鴛鴦也! 其要欲使人真能自繡鴛鴦而已.

徐光啟 · 幾何原本雜議 (1607)

"杜氏三術" (in modern terminology)
(杜德美 [Pierre Jartoux], 1701)

$$\pi = 3[1 + 1^2/(4.\ 3!) + (1^2.\ 3^2)/(4^2.\ 5!) + (1^2.\ 3^2.\ 5^2)/(4^3.\ 7!) + \cdots]$$

(Isaac Newton, 1676)

Isaac Newton
(1642-1767)

$$r \sin(a/r) = a - a^3/(3!\, r^2) + a^5/(5!\, r^4) - a^7/(7!\, r^6) + \cdots$$

$$r \operatorname{vers}(a/r) = a^2/(2!\, r) - a^4/(4!\, r^3) + a^6/(6!\, r^5) - \cdots$$

(James Gregory, 1667)

James Gregory
(1638-1675)

$$\pi = 3[1 + 1^2/(4.\,3!) + (1^2.\,3^2)/(4^2.\,5!) + (1^2.\,3^2.\,5^2)/(4^3.\,7!) + \cdots]$$

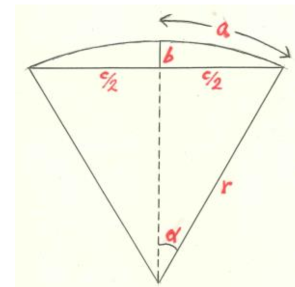
$$r \sin(a/r) = a - a^3/(3!\,r^2) + a^5/(5!\,r^4) - a^7/(7!\,r^6) + \cdots$$

“杜氏三術”

$$r \operatorname{vers}(a/r) = a^2/(2!\,r) - a^4/(4!\,r^3) + a^6/(6!\,r^5) - \cdots$$

$$c = 2r \sin\alpha = 2a - (2a)^3/(4.3!\,r^2) + (2a)^5/(4^2.5!\,r^4) - (2a)^7/(4^3.7!\,r^6) + \cdots$$

$$b = r \operatorname{vers}\alpha = (2a)^2/(4.2!\,r) - (2a)^4/(4^2.\,4!\,r^3) + (2a)^6/(4^3.\,6!\,r^5) - (2a)^8/(4^4.\,8!\,r^7) + \cdots$$



$$2a = c + (1^2.\,c^3)/(4.3!\,r^2) + (1^2.\,3^2.\,c^5)/(4^2.5!\,r^4) + (1^2.3^2.\,5^2.\,c^7)/(4^3.7!\,r^6) + \cdots$$

$$a = r\sin\alpha + (1^2)(r\sin\alpha)^3/(3!\,r^2) + (1^2.\,3^2)(r\sin\alpha)^5/(5!\,r^4) + \cdots$$

$$a^2 = 2r^2 \operatorname{vers}\alpha + (2.1^2)(2r\operatorname{vers}\alpha)^2/(4!) + (2.1^2.2^2)(2r\operatorname{vers}\alpha)^3/(6!\,r) + \cdots$$

“杜氏九術”,
應為“明氏六術”

$$(2a)^2 = (8b)r + 2.1^2(8b)^2/(4.\,4!) + (2.1^2.2^2)(8b)^3/(4^2.\,6!\,r) + \cdots$$

《割圆密率捷法》序

昔元[7]家藏钞本《割圆捷法》一帙，不知为何人之书，故《畴人传》[8]未载。今致仕归扬州，读天长岑氏绍周所校刻《割圆密率捷法》四卷，及甘泉罗氏茗香[9]跋，始知是书为满洲明静庵先生撰于乾隆之时。

盖自八线表[10]成，推算有成数，而未发其理。墨守者谁复推其所以然？此书则以己意悟明其法，任求何边之数，不过几次乘除，一二时即可得之，真步天捷法也！罗氏又欲补撰《畴人传》，叙述宋元以来精心求天圆而实事求是之人，于秦[11]、李[12]、朱[13]、赵[14]及　本朝明[15]、陈[16]诸公，接补为传，使四元[17]诸法学者得而习之，不其伟欤！

夫天西洋人来于明末，乘诸古法失传之时，所以有功于天学；迨及末流，多习天主邪教，惑诱为害，所以　命其回国[18]。若使今之人益明古法，不但有所接续，且使西法不得擅为秘术。庶几中土之书，明明布列；步天之士，蔼蔼周行，是所望也。

道光二十年(1840)正月　节性斋老人阮元序

《割圓密率捷法譯注》, 明安圖原著; 羅見今譯注, 內蒙古教育出版社, 1998.

《割圆密率捷法》序

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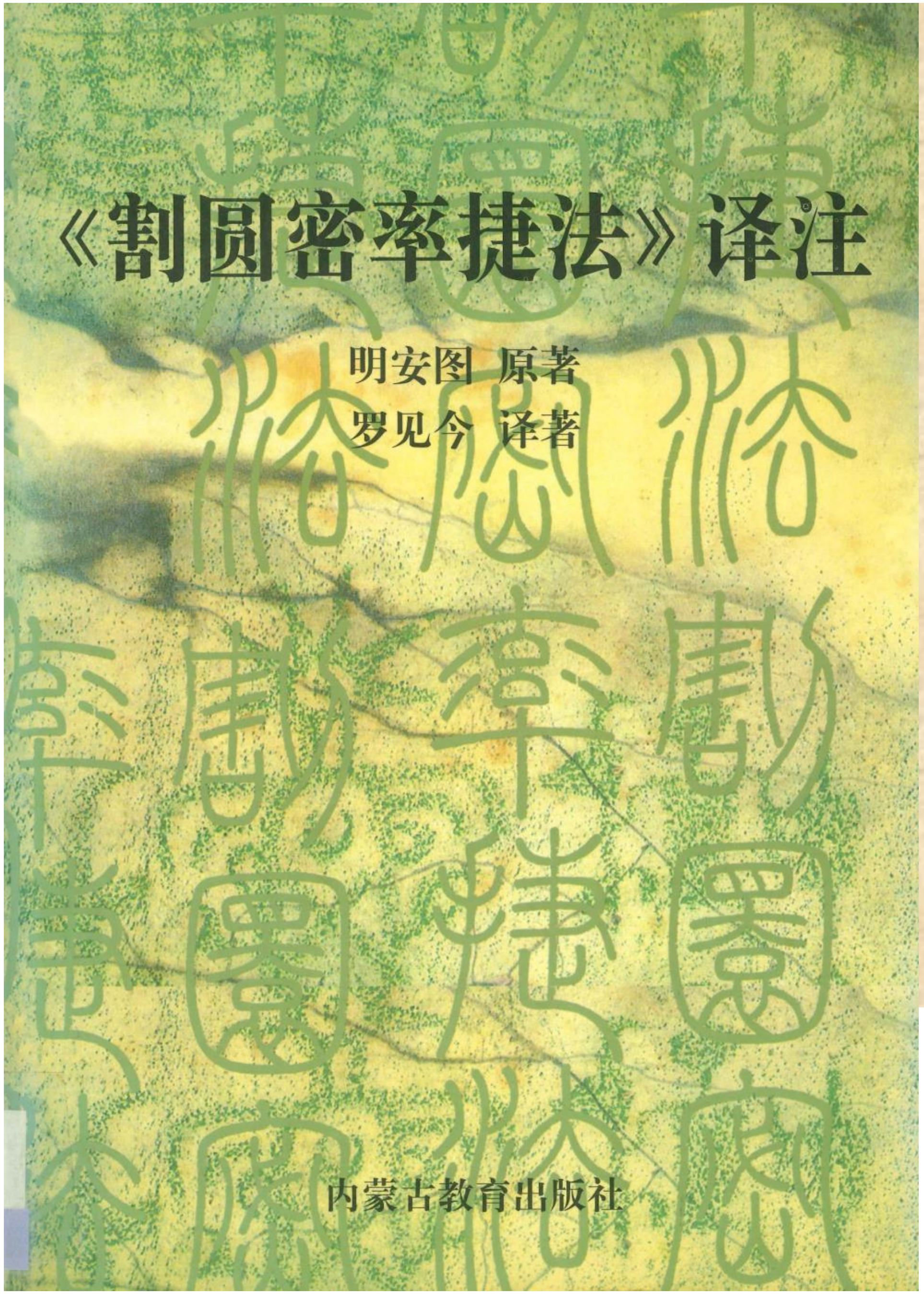
Independent derivation without knowledge of calculus, which was not yet transmitted into China at the time!

Through the *Ge Yuan Mi Lű Jie Fa*, Chinese mathematics during the 18th century and the first part of the 19th century appears to have been a lively and productive discipline in which the Chinese did much more than merely assimilate what they had been taught by the Jesuits; they continued in the directions determined by that teaching. It maintains a separate identity, an independence characterized by its being a synthesis between Western and traditional Chinese elements.

Catherine Jami, Western influence and Chinese tradition in an eighteenth century Chinese mathematical work, *Historia Mathematica*, 15 (1988), 311-331.

Catherine Jami, *Etude du livre "Méthodes rapides de trigonométrie et du rapport précis du cercle" de Ming Antu* (*?-1765?*), Thèse de doctorat, Université Paris XIII, 1985.

Catherine Jami
詹嘉玲



《割圓密率捷法》
譯注

明安圖著

羅見今譯著

[Quick Methods for Trigonometry and for Determining the Precise Ratio of the Circle]

1998

Luo Jian-jin and friend, Sanya, China, 2016 March.

但是, 這種交流並非一帆風順. 1723年雍正皇帝即位後, 由於宗教矛盾的積累, 改變了康熙的做法, 採納浙閩總督滿寶的建議, 把外國傳教士除在欽天監任職者外, 一律驅趕到澳門, 不許擅入內地, 開始了近代百餘年閉關自守的歷史. 西方科學向中國輸入基本停止, 對國內士大夫加強思想統制, 興文字獄, 原來不承認落後的"西學中源說"、有意學習西方的"匯通中西說"開始轉向漢學, 校勘注釋古書, 以後形成了乾嘉學派. 另一方面, 18世紀初羅馬教皇頒佈的教令對傳教方式嚴加限制, 傳入的外文書多未能譯成漢語, 教士的主要目的本來就在於傳教, 他們也不是較高水準的科學家. 所有這些形成了明安圖學術工作的封閉背景, 他雖處在高層, 也不能不受到嚴重影響.

事實上, 當時西方一些先進的科學, 例如太陽系學說、萬有引力定律等, 由於宗教的排斥, 不可能傳入中國. 傳進來的知識也存在不完整、不系統的缺陷. 在數學上，繼韋達（F. Vieta, 1540-1603, 法國人）的符號代數之後, 笛卡兒（René Descarters, 1596-1650, 法國人）的解析幾何和牛頓（Isaac Newton, 1643-1727,英國人）、萊布尼茲（G. W. Leibniz, 1646-1716, 德國人）的微積分均已問世, 宣告了數學新時代的來臨. 但這些科學知識中國人基本上是不知道的. 明安圖的大半生, 實際上不得不在閉關的黑暗中摸索, 尋找前進的道路, 從傳統數學中獨闢蹊徑, 希望能夠進入數學發展的主流.

在中國, 清代數學家從明安圖開始, 董祐誠、項名達、徐有壬、戴煦、李善蘭、鄒伯奇、夏鸞翔、華蘅芳等人, 主要也為了研究三角函數、對數和有限差分, 對級數產生了濃厚的興趣. 他們使用的方法基本上是傳統的. 由於傳統數學沒有形成完整的符號體系和演繹系統, 妨礙了它的發展, 與近代數學脫節, 沒有進入微積分的發展時期. 但是, 正如文章開始所說, 中算家們發揮了自己的長處, 應用離散的手段, 處理連續的物件, 在與世隔絕的環境中, 仍然引入了一些有價值的思想和方法, 戴煦的工作即是一例.

第12章 徐有壬、戴煦正切數研究的領先成果

羅見今,《中算家的計數論》, 科學出版社, 2022.

在中國, 清代數學家從明安圖開始, 董祐誠、項名達、徐有壬、戴煦、李善蘭、鄒伯奇、夏鸞翔、華蘅芳等人, 主要也為了研究三角函數、對數和有限差分, 對級數產生了濃厚的興趣. 他們使用的方法基本上是傳統的. 由於傳統數學沒有形成完整的符號體系和演繹系統, 妨礙了它的發展, 與近代數學脫節, 沒有達到微積分的發展時期. 但是, 正如文章開始所說, 中算家們發揮了自己的長處, 運用離散的手段, 處理連續的物件, 在與世隔絕的環境中, 仍然引入了一些有價值的思想和方法, 戴煦的工作即是一例.

- 明安圖 (c. 1692-1763)
- 項名達 (1789-1850)
- 董祐誠 (1791-1823)
- 徐有壬 (1800-1860)
- 戴煦 (1805-1860)
- 李善蘭 (1811-1882)
- 鄒伯奇 (1819-1869)
- 夏鸞翔 (1823-1864)
- 華蘅芳 (1833-1902)

李儼, 三角術及三角函數表之東來, *科學雜誌* 12(10) (1927), 1245-1393.

李儼, 明清算家之割圓術研究, *科學雜誌* 12(11) (1927), 1487-1520; 12(12) (1927), 1721-1766; 13(1) (1928), 53-102; 13(2) (1928), 200-250.

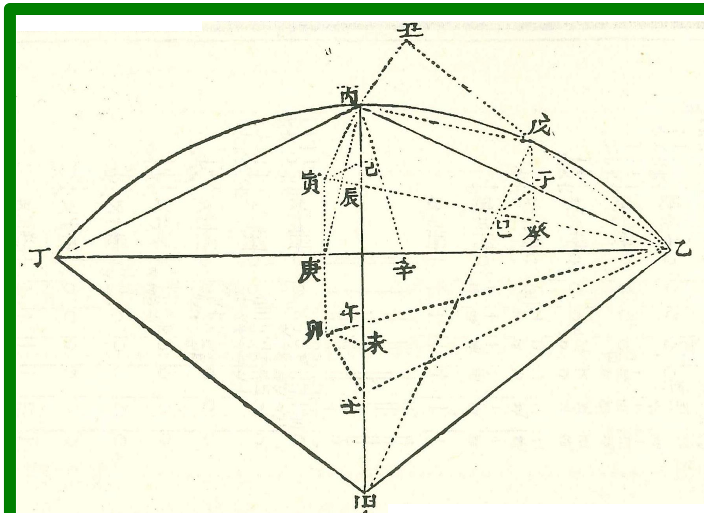
李儼, 中算家之級數論, *科學雜誌* 13(9) (1929), 1139-1172; 13(10) (1929), 1349-1401.

Li Yan [李儼]
(1892-1963)

Chen Jiang-ping [陳建平], Trigonometric tables: explicating their construction principle in China, *Archive for History of Exact Sciences*, 69 (5) (2015), 491-536.

明安圖在《割圜密率捷法》卷三中求 sin 2θ 的展開式時發現了 Catalan Numbers (Leonhard Euler, 1751; Eugène Charles Catalan, 1838).

明安圖使用的方法, 獨闢蹊徑, 構造一個幾何圖形, 應用連比例法來求展開式, 與西方的和現代的方法都不相同.



「在運算中出現卡塔蘭數既不是偶然和巧合, 也不是明安圖無意中碰到的, 而是他經過精心設計, 只有在他熟練掌握這些級數和卡塔蘭數的性質時, 才可能得心應手將其展示出來. 當然, 傳統數學在18 世紀尚未引入近代數學體系, 在符號系統、表達方式等方面存在差距. 因此, 不熟悉歷史的人認為中算沒有證明; 需要對“寓理於算” 的道理深入理解, 需要依據原著, 對算式進行具體分析, 才能取得正確的認識.」

(羅見今,《中算家的計數論》, 2022.)

One interesting result by Ming Antu:

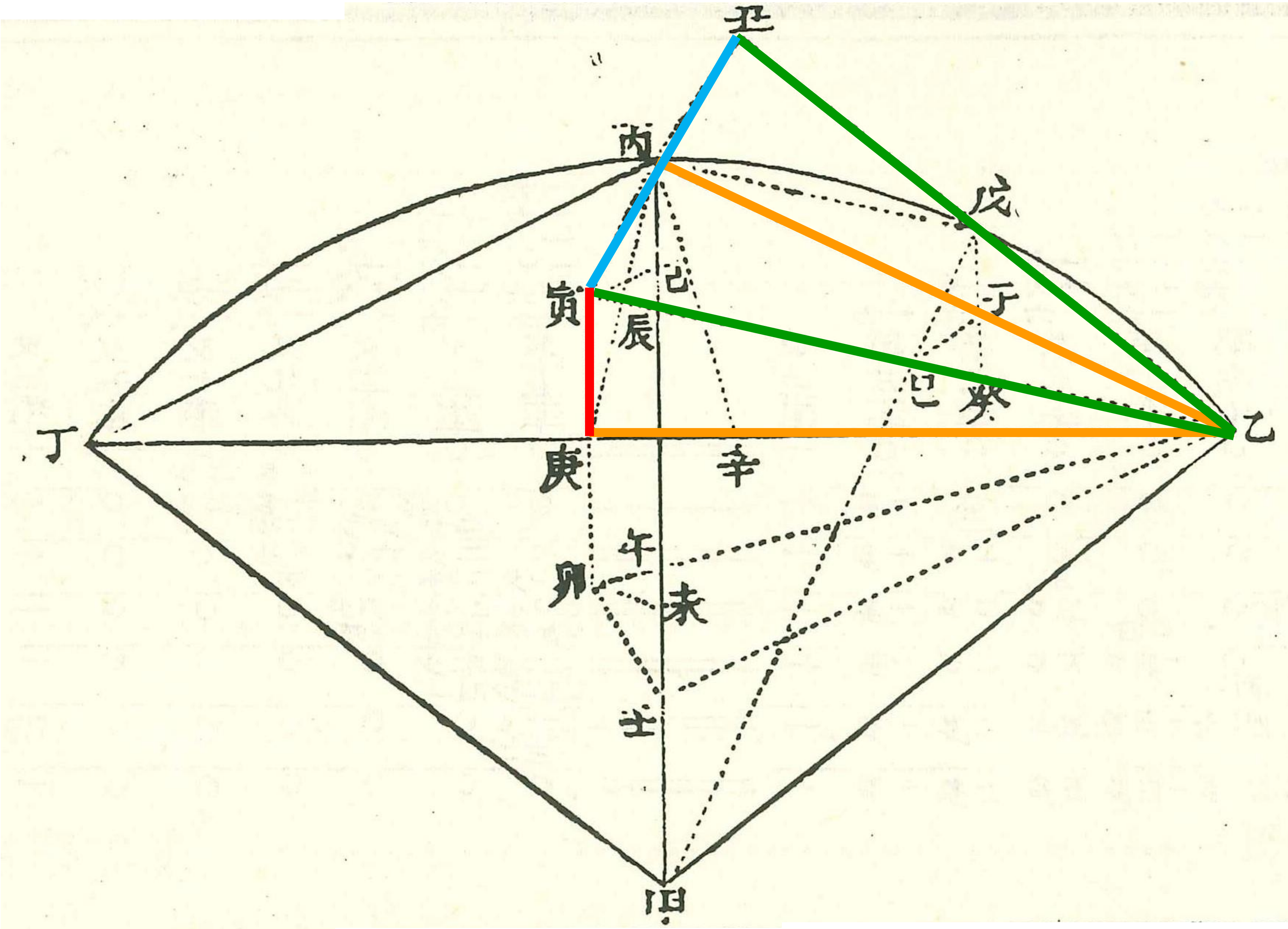
$$\sin 2\theta = 2s - C_1 s^3/4^0 - C_2 s^5/4^1 - C_3 s^7/4^2 - C_4 s^9/4^3 - \cdots$$

where $s = \sin\theta$, and $C_i = i^{\text{th}}$ **Catalan Number**

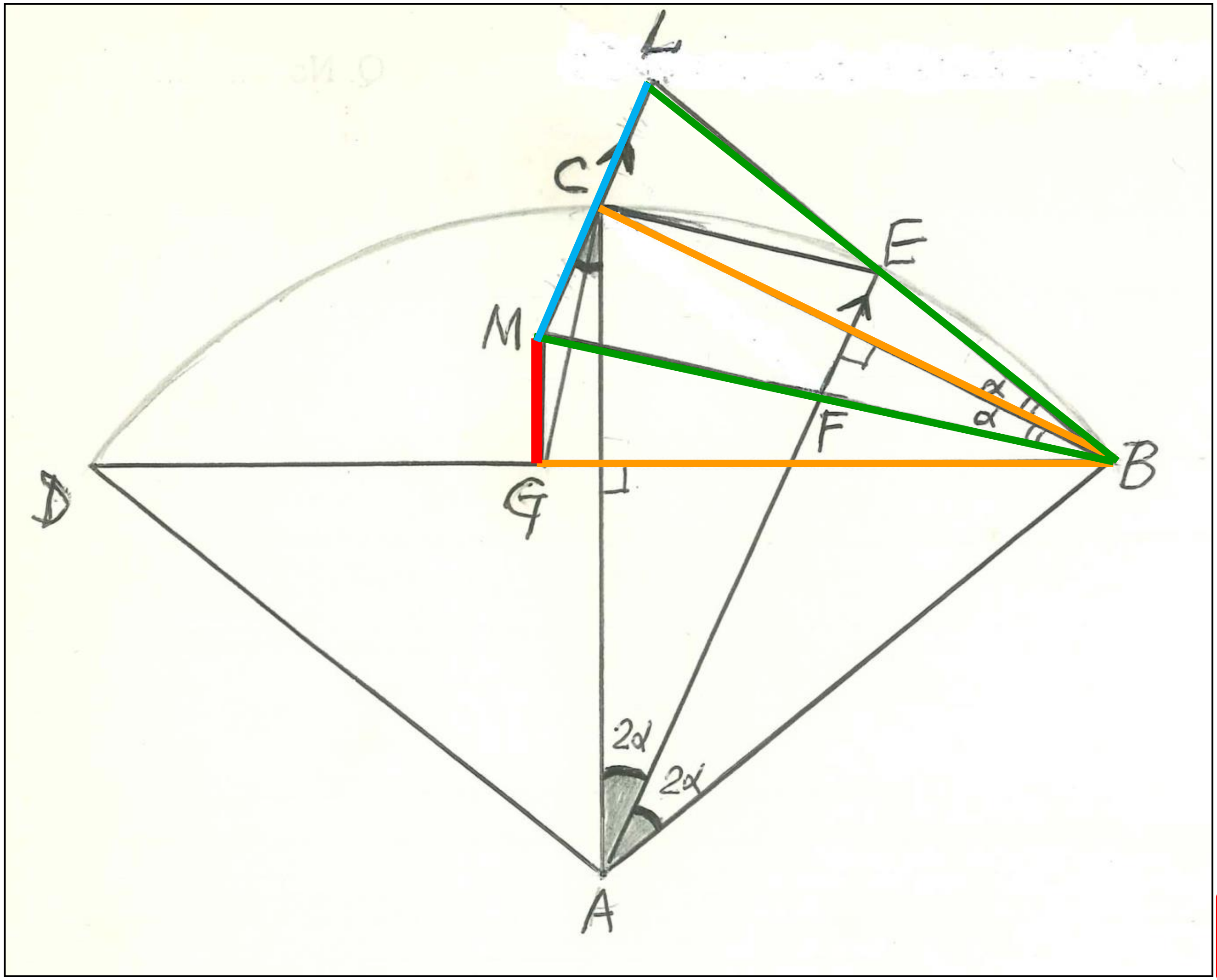
$$= {}_{2i}C_i / (i+1).$$

$C_1 = 1$, $C_2 = 1$, $C_3 = 2$, $C_4 = 5$, $C_5 = 14$, $C_6 = 42$, $C_7 = 132$, $C_8 = 429$, $C_9 = 1430$, $C_{10} = 4862$, … .

- **P. J. Larcombe, The 18th century Chinese discovery of the Catalan Numbers, *Mathematical Spectrum*, 32 (1) (1999/2000), 5-7.**
- 羅見今, 明安圖是卡塔蘭數的首創者, *內蒙古大學學報. 自然科學版*, 19 (2) (1998), 239-245.



明安圖《割圜密率捷法》• 卷三

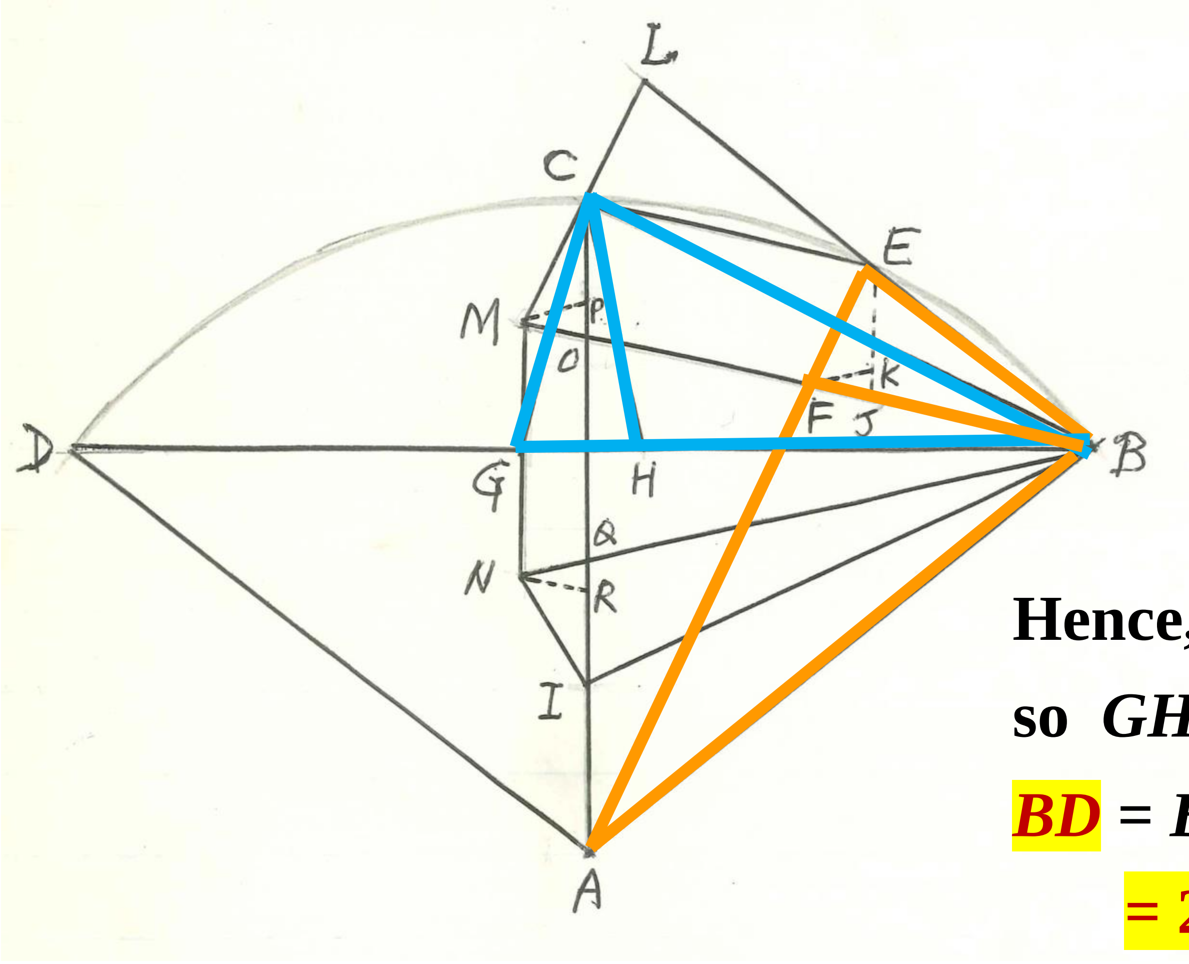


AC ⊥ BD.
Make *BC = BG*. Let the angle bisector of ∠*BAC* cut the circular arc at *E*.
Join *BE* and produce to make *BE = LE*.
Join *LC* and produce to make *LC = MC*.
Note that *LM* // *EF* (by Midpoints Theorem).
So, Δ *BLC* ≅ Δ *BMC*.

Hence, ∠*MCA* = ∠*EAC* = 2*α*.

Note that ∠*ACB* = ∠*MCB* − ∠*MCA* = *π*/2 − 2*α*.

Hence, ∠*MBG* = *α*, Δ *BMC* ≅ Δ *BMG*, and MG ⊥ *BG*.



Make *BG* = *BC* and *DH* = *DC*.

Δ *ABE* ~ Δ *BCG*.

Δ *ABE* ~ Δ *BEF* and

Δ *BCG* ~ Δ *CGH*.

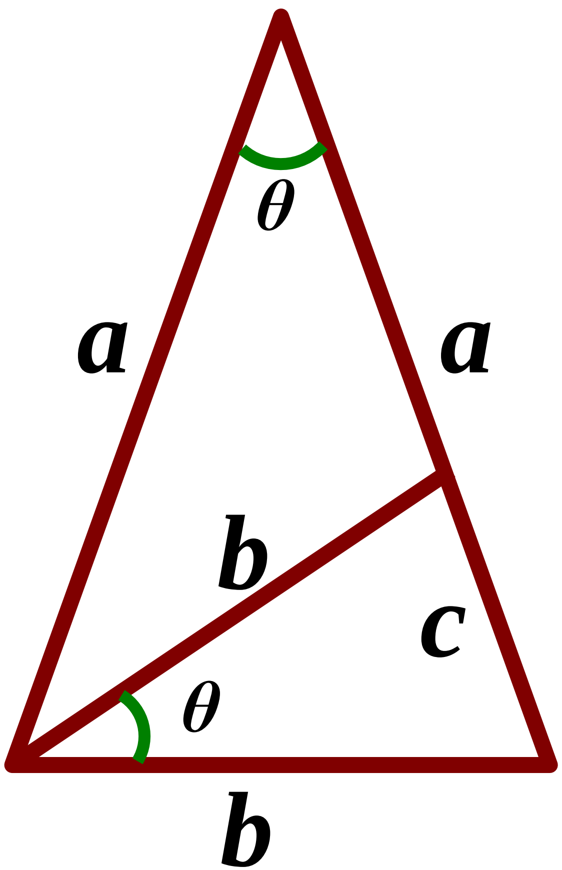
Hence, *AB* : *EF* = *BC* : *GH*,

so *GH* = (*BC*. *EF*) / *AB*.

BD = *BC* + *DC* – *GH*

= 2*BC* – (*BC*. *EF*) / *AB*.

Basic Lemma (Continued Proportion of sides of an isosceles triangle)



$$a : b = b : c$$

or

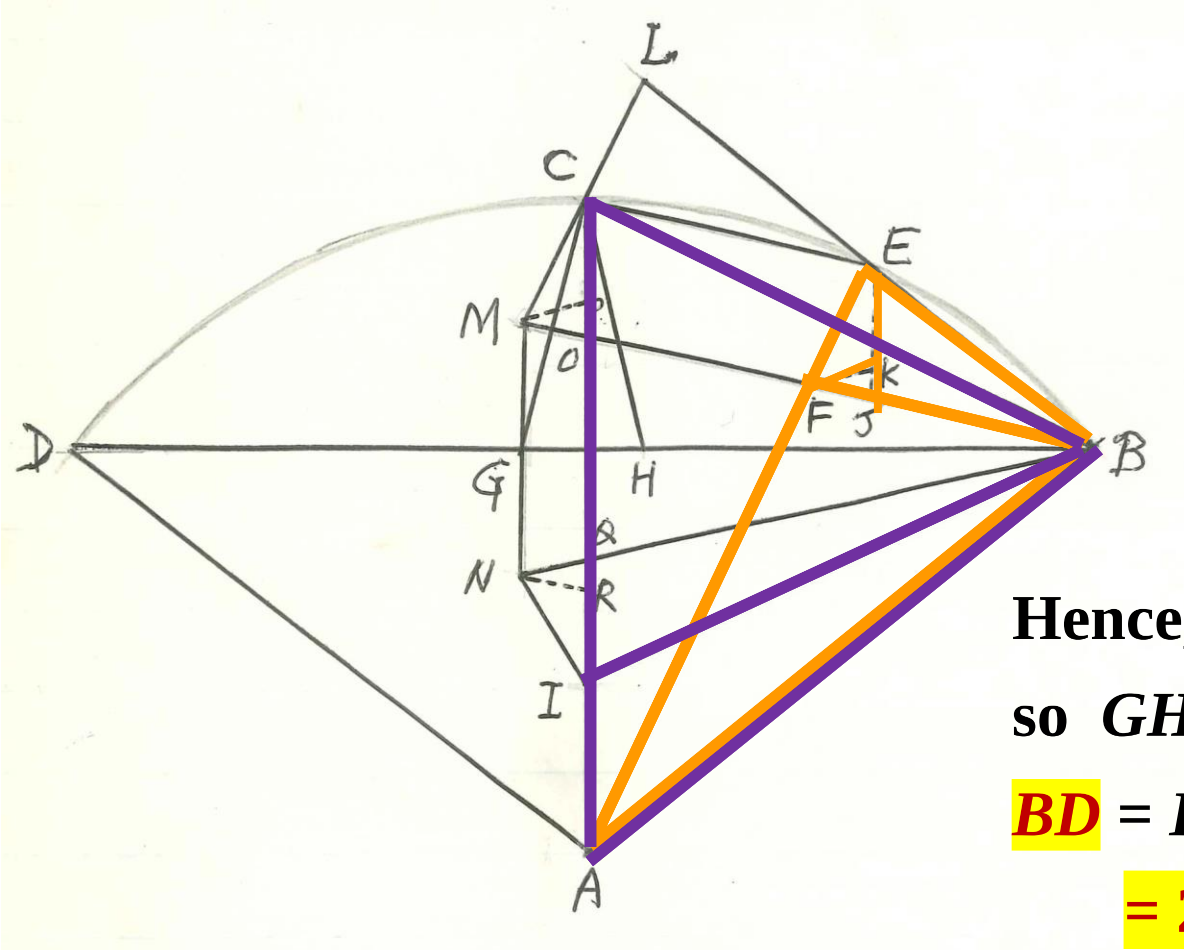
$$b^2 = ac$$

Hence,

$$a : b : c = 1 : x : x^2$$

[...] 以半徑為連比例第一率 [...]

明安圖《割圜密率捷法》



Make $BG = BC$ and
$DH = DC$.

$\Delta\, ABE \sim \Delta\, BCG$.

$\Delta\, ABE \sim \Delta\, BEF$ and

$\Delta\, BCG \sim \Delta\, CGH$.

Hence, $AB : EF = BC : GH$,

so $GH = (BC.\ EF) / AB$.

$BD = BC + DC - GH$

$= 2BC - (BC.\ EF) / AB$.

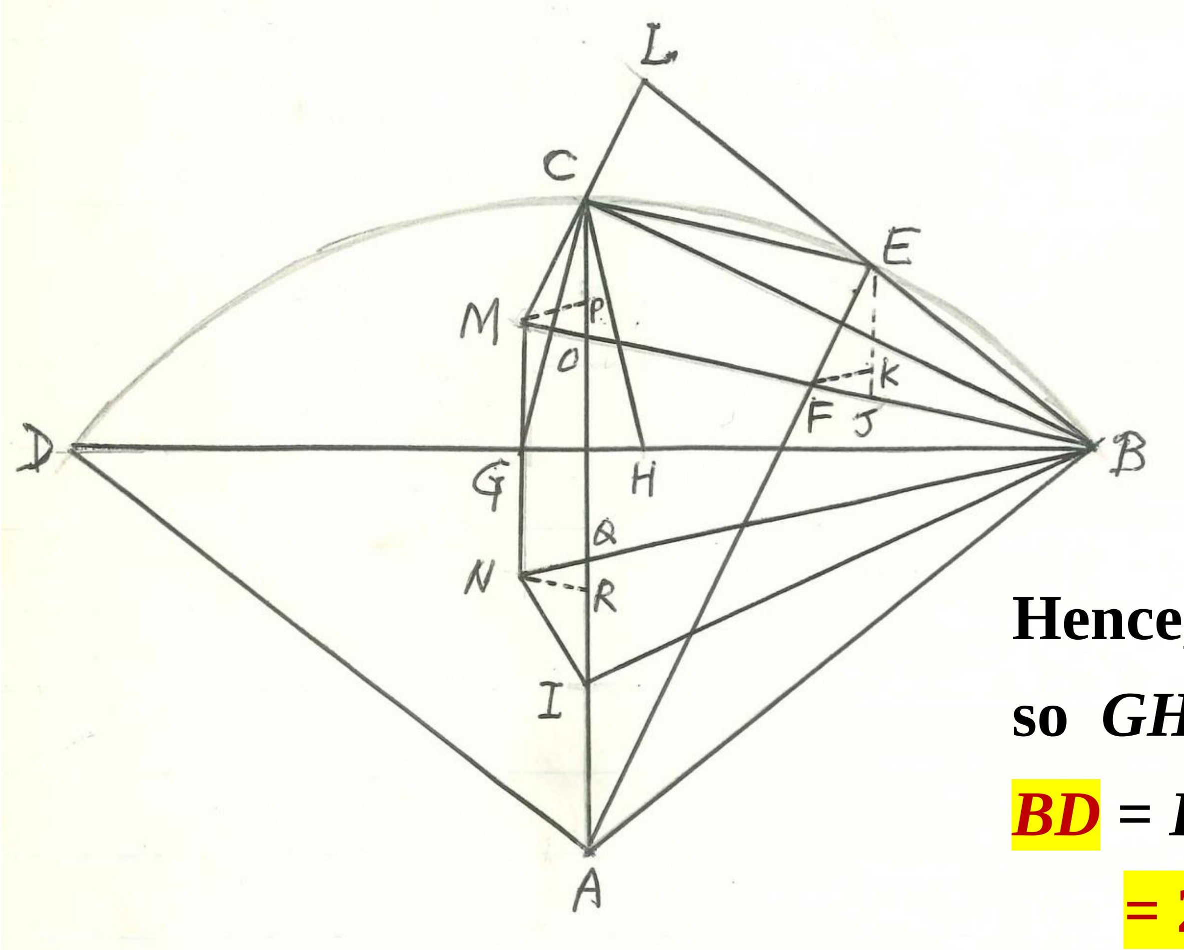
$\Delta\, ABC \sim \Delta\, BCI$.

Hence, $AB : BC = BC : CI$, so $CI = x^2$ where $BC = x$.

$AB : BC : CI = 1 : x : x^2$.

$\Delta\, ABE \sim \Delta\, BEF \sim \Delta\, EFJ \sim \Delta\, FJK$.

Hence, $AB : BE : EF : FJ : JK = 1 : p : p^2 : p^3 : p^4$ where $BE = p$.



Make $BG = BC$ and $DH = DC$.

$\Delta\, ABE \sim \Delta\, BCG$.

$\Delta\, ABE \sim \Delta\, BEF$ and

$\Delta\, BCG \sim \Delta\, CGH$.

Hence, $AB : EF = BC : GH$,

so $GH = (BC.\, EF) / AB$.

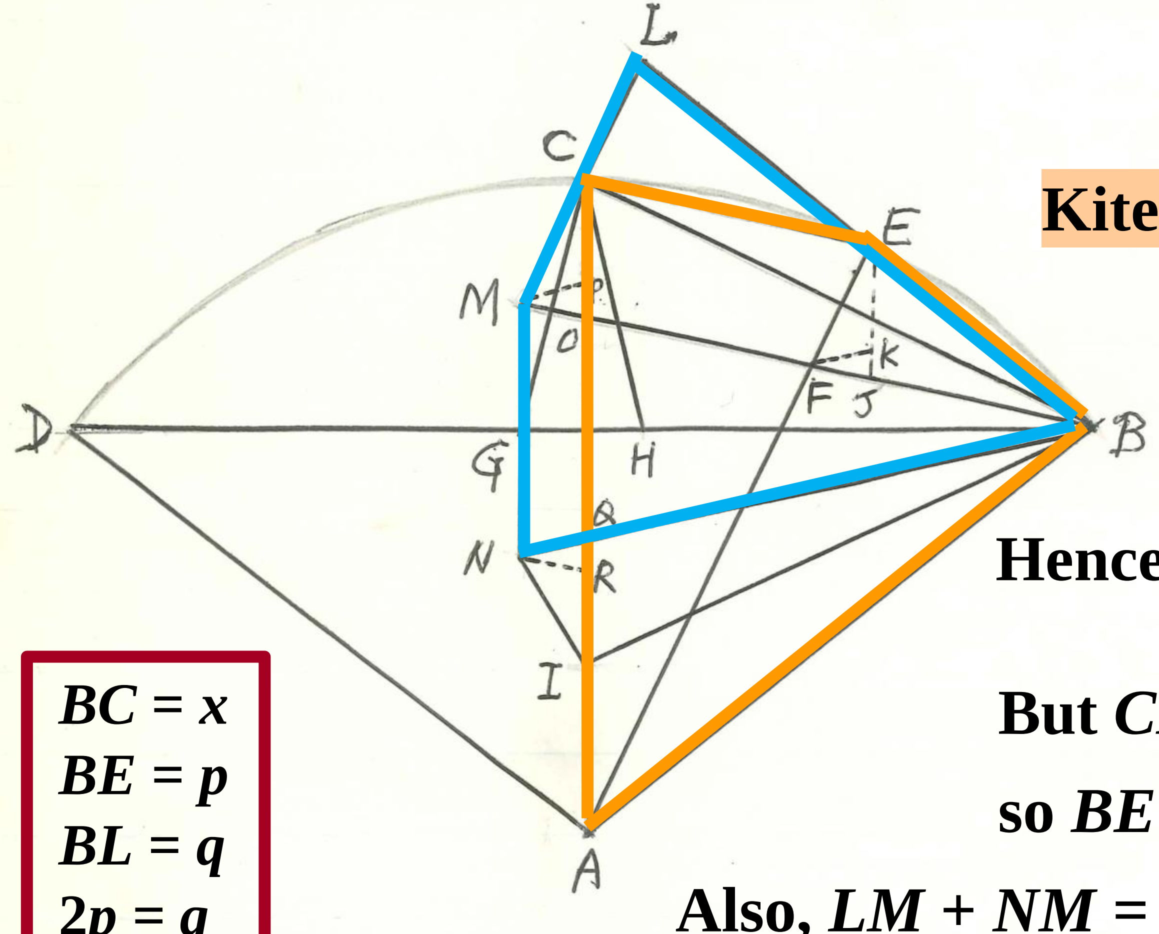
$BD = BC + DC - GH$

$= 2BC - (BC.\, EF) / AB$.

$AB : BC : CI = 1 : x : x^2$ where $BC = x$.

$AB : BE : EF : FJ : JK = 1 : p : p^2 : p^3 : p^4$ where $BE = p$.

Hence, $BD = 2x - x\,p^2 = 2x - x\,q^4/4$ where $q = 2p = BL$.



$BC = x$
$BE = p$
$BL = q$
$2p = q$

Kite $ABEC$ ~ Kite BLMN.

Hence, $AB : (BE + CE)$
$= BL : (LM + NM)$.

But $CE = BE = LE$,

so $BE + CE = BL$.

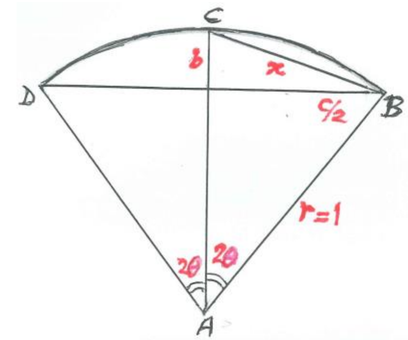
Also, $LM + NM = CI + JK$ (Exercise)

Therefore, $AB : BL = BL : (CI + JK)$,

so $AB : BL : (CI + JK) = 1 : q : q^2$.

Therefore, $x^2 = CI = (CI + JK) – JK = q^2 – p^4$.

Hence, we have $x^2 = q^2 – q^4 /16$, since $q = 2p$.



$BD = 2\sin 2\theta$
$BC = 2\sin\theta$
$r = 1$

$$x^2 = q^2 - q^4/4^2.$$

$$x^4 = x^2 . x^2 = q^4 - 2q^6/4^2 + q^8/4^4.$$

$$x^4/4^2 = q^4/4^2 - 2q^6/4^4 + q^8/4^6.$$

$$x^6 = x^2 . x^4 = q^6 - 3q^8/4^2 + 3q^{10}/4^4 - q^{12}/4^6.$$

$$2x^6/4^4 = 2q^6/4^4 - 6q^8/4^6 + 6q^{10}/4^8 - 2q^{12}/4^{10}.$$

$$x^8 = x^2 . x^6 = q^8 - 4q^{10}/4^2 + 6q^{12}/4^4 - 4q^{14}/4^6 + q^{16}/4^8.$$

$$5x^8/4^6 = 5q^8/4^6 - 20q^{10}/4^8 + 30q^{12}/4^{10} - 20q^{14}/4^{12} + 5q^{16}/4^{14}.$$

etc.

Hence, $x^2 + x^4/4^2 + 2x^6/4^4 + 5x^8/4^6 + \dots = q^2$,

or $q^2 = C_1 x^2 + C_2 x^4/4^2 + C_3 x^6/4^4 + C_4 x^8/4^6 + \dots$

$= (4/x)\,[\, C_1 x^3/4 + C_2 x^5/4^3 + C_3 x^7/4^5 + C_4 x^9/4^7 + \dots]$,

or $xq^2/4 = C_1 x^3/4 + C_2 x^5/4^3 + C_3 x^7/4^5 + C_4 x^9/4^7 + \dots$,

or $xp^2 = x(q/2)^2 = C_1 x^3/4 + C_2 x^5/4^3 + C_3 x^7/4^5 + C_4 x^9/4^7 + \dots = GH$.

$BD = 2BC - GH = 2x - C_1 x^3/4 - C_2 x^5/4^3 - C_3 x^7/4^5 - C_4 x^9/4^7 - \dots$.

i.e. $\sin 2\theta = 2s - C_1 s^3/4^0 - C_2 s^5/4^1 - C_3 s^7/4^2 - C_4 s^9/4^3 - \dots,\ s = \sin\theta = x/2.$

一六一六一六一六一六一六 又十五率	一六一六一六一六一六 又十三率	一六一六一六一六 又十一率	一六一六一六 又九率	一六一六 又七率	一六 又五率	又三率
○	○	○	○一多	○二少	一少一一	一
○	○三少	○六多少	一多六少	二少二二	○	一
○二○少	三少二○多	六多二○少	五少五五	○	○	一
二○少一四○多	二八多七○少	一四少一四	○	○	○	一
一二○多二五二少	四二少四二	○	○	○	○	一
一三二少一三二	○	○	○	○	○	一
○	○	○	○	○	○	一

一六一六一六一六一六一六 第十五率	一六一六一六一六一六 第十三率	一六一六一六一六 第十一率	一六一六一六 第九率	一六一六 第七率	一六 第五率	第三率
○	○	○	○	○	○一	一
○	○	○	○	○二	一多	一
○	○	○	○五	二多	一多	一
○	○	○一四	五多	二多	一多	一
○	○四二	一四多	五多	二多	一多	一
○一三二	四二多	一四多	五多	二多	一多	一
一三二多	四二多	一四多	五多	二多	一多	一

率之一少五次十六分十三率之五
多六次十六分十五率之十與四次
十六分十一率之一二得數爲同母
相等次另書十一率同母相等數于
上三率相等數于下乘除得五次十
六分十三率之一少六次十六分十
五率之六與五次十六分十三率之
一二得數爲同母相等次另書十三

明安圖《割圜密率捷法》卷三

中學課堂三角學教學中值得思考的一些問題

- 使用直角三角形邊長比例與使用單位圓旋轉角的方法之間的比較?
- 明清時代從西方傳入引進的“八線圖”有何啟發?
- 何時在學校課程中引入三角學?
- 如何選擇及整合三角學與其他數學課題?

三角函數所起的作用, 不僅僅是作為
一種“速記語言”以化簡論證中
需要書寫的文字,
它的角色遠不止於此.

儘管正弦定理和餘弦定理
均以幾何為基礎, 它們在給定三角形
某些邊和角的條件下, 確定三角形
的大小和形狀, 大展拳腳, 表現出色.

在高等數學中, 這些三角函數
更加突顯它們作為函數
所發揮的作用.

三角學的運用

初等數學：

- 計算圖形的邊長、角度和面積
- 測量問題
- 三維問題

高等數學：

- 微積分學習的準備知識
- 協助研究在其他學科出現的周期性現象

三角學的運用

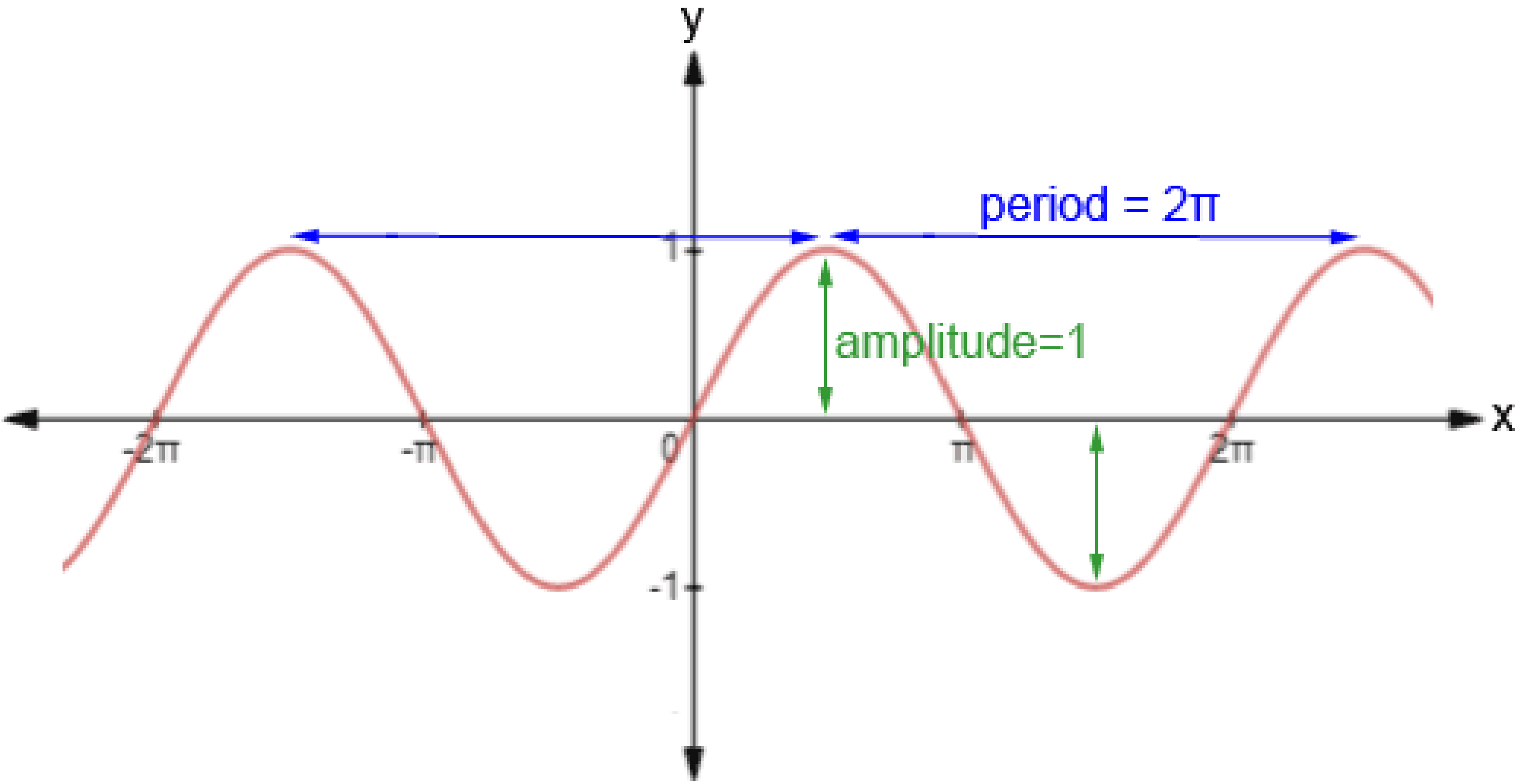
初等數學：

- 計算圖形的邊長、角度和面積
- 測量問題
- 三維問題

這是人類研究自然現象中發展出來的絕妙思想，由來已久！

高等數學：

- 微積分學習的準備知識
- 協助研究在其他學科出現的周期性現象



sin: $\mathcal{R} \rightarrow \mathcal{R}$ is a periodic function on the real line $\mathcal{R}$ with period 2π (radian).

x	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6
0°	0·0000	0017	0035	0052	0070	0087	0105
1	·0175	0192	0209	0227	0244	0262	0279
2	·0349	0366	0384	0401	0419	0436	0454
3	·0523	0541	0558	0576	0593	0610	0628
4	·0698	0715	0732	0750	0767	0785	0802
5	0·0872	0889	0906	0924	0941	0958	0976
6	·1045	1063	1080	1097	1115	1132	1149
7	·1219	1236	1253	1271	1288	1305	1323
8	·1392	1409	1426	1444	1461	1478	1495
9	·1564	1582	1599	1616	1633	1650	1668

Why we like to use radian measure in mathematics at a higher level? ?

x	Deg.	sin x
r	°	
0·00	0·00	0
·01	0·57	·0100
·02	1·15	·0200
·03	1·72	·0300
·04	2·29	·0400
0·05	2·86	0·0500
·06	3·44	·0600
·07	4·01	·0699
·08	4·58	·0799
·09	5·16	·0899
0·10	5·73	0·0998
·11	6·30	·1098
·12	6·88	·1197
·13	7·45	·1296
·14	8·02	·1395

$\mathcal{R}$ is the set of real numbers.

sin : $\mathcal{R} \to \mathcal{R}$

sin x = sine of angle of measure x (in radian)

SIND : $\mathcal{R} \to \mathcal{R}$

SIND x = sine of angle of measure x (in degree)

sin x = SIND$(180x/\pi)$

SIND x = sin$(\pi x/180)$

Can the Taylor series expansion produce good approximation to all kinds of functions? What can be done if it does not ?

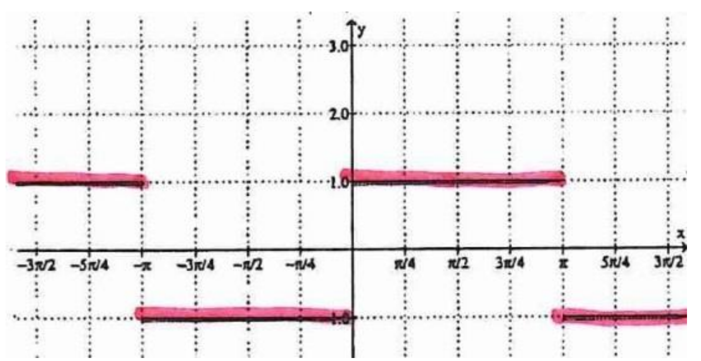
For instance, how can we handle a function like

$f(x) = 1$ on $[k\pi, (k+1)\pi]$

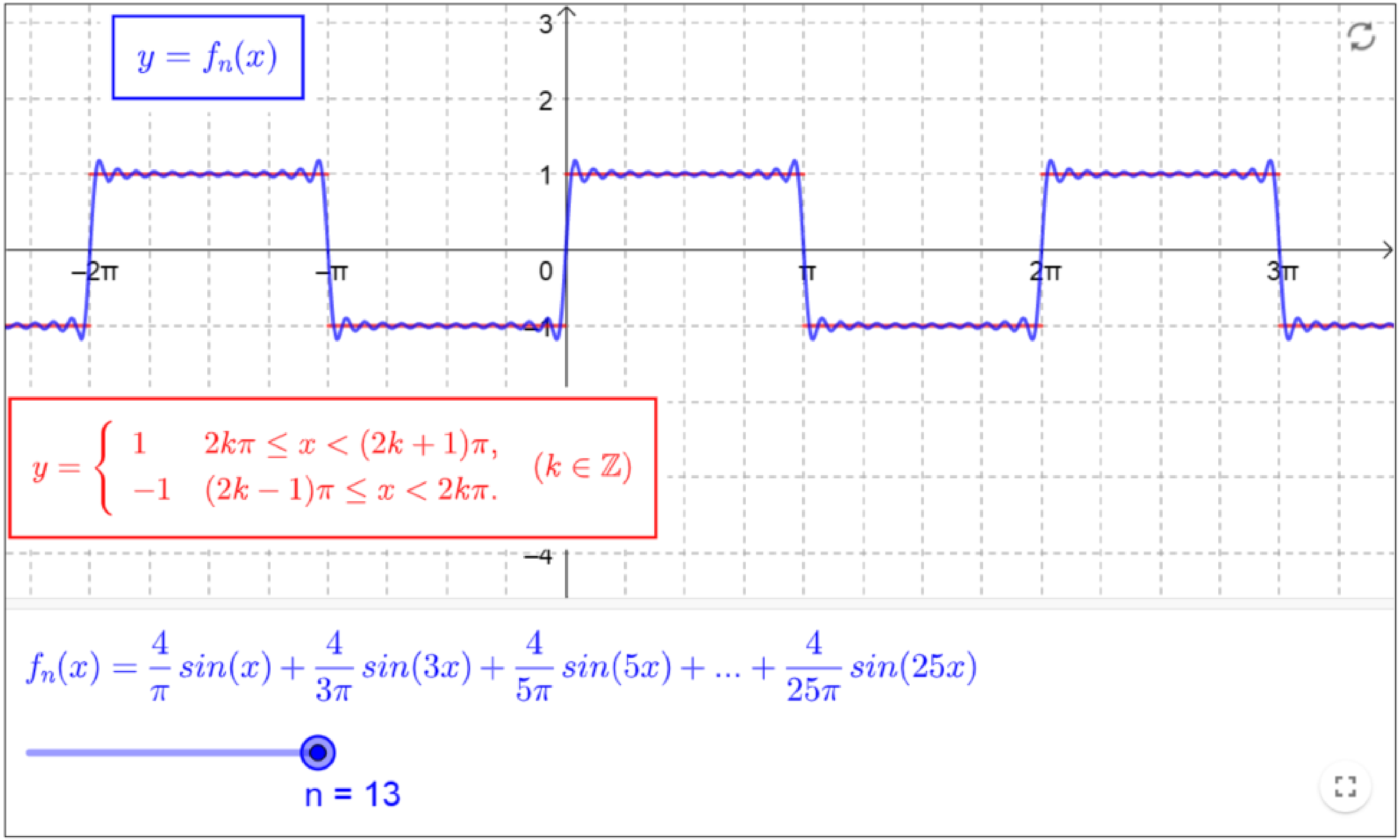
if k is …, -6, -4, -2, 0, 2, 4, 6, … ;

$f(x) = -1$ on $[k\pi, (k+1)\pi]$

if k is …, -5, -3, -1, 1, 3, 5, 7, … ?



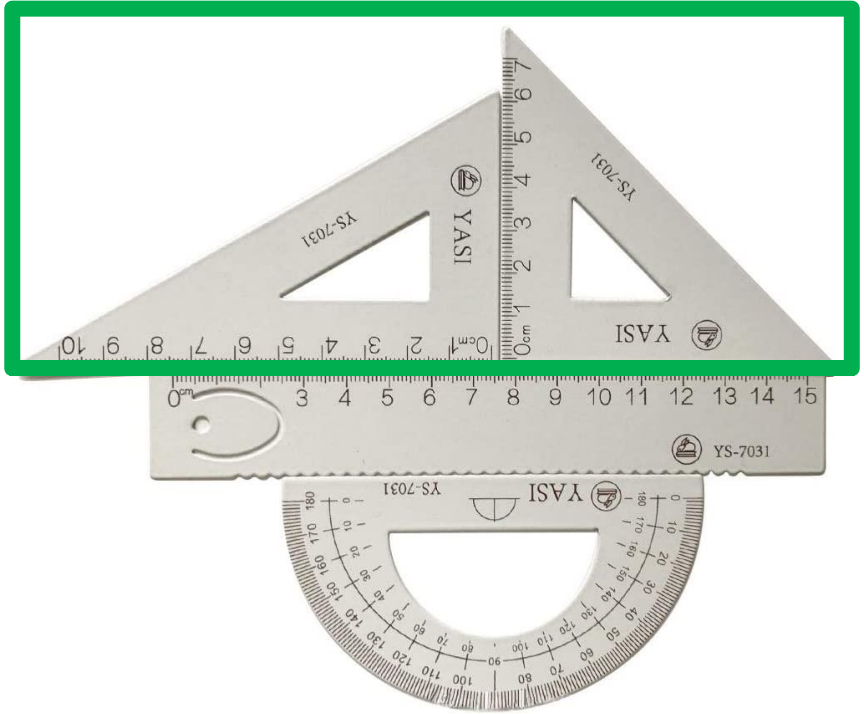
Fourier series expansion



https://ggbm.at/9859942

讓我以一個中學時代的課堂小故事作結,
它展示了三角學與其他數學課題的一些關連.

一副 (兩塊)
標準三角尺



身為中學生, 我每天都帶著這副“架生”上學. 但現在回想起來, 我幾乎從未使用過它!
如果我需要構作垂直線或平行線, 我要麼使用圓規和直尺, 要麼 (如果不必考慮準確性)徒手繪製. 如果我需要繪製一定尺寸的角度, 我會使用量角器.

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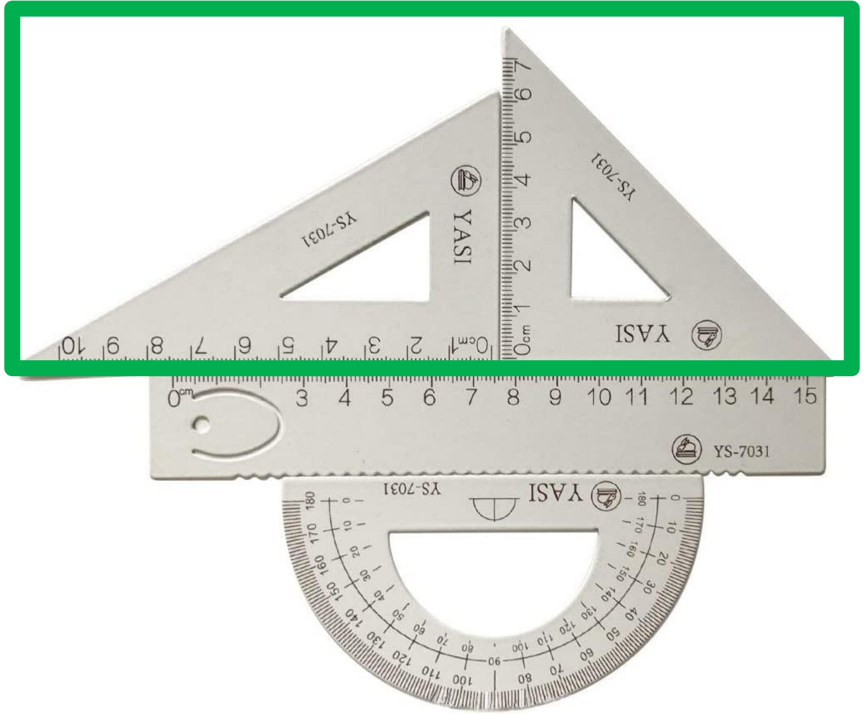

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那麼, 對這副每個學生都帶著上學的“架生”,
還能說些什麼呢? (現在的學生, 還會帶著它上學嗎?)

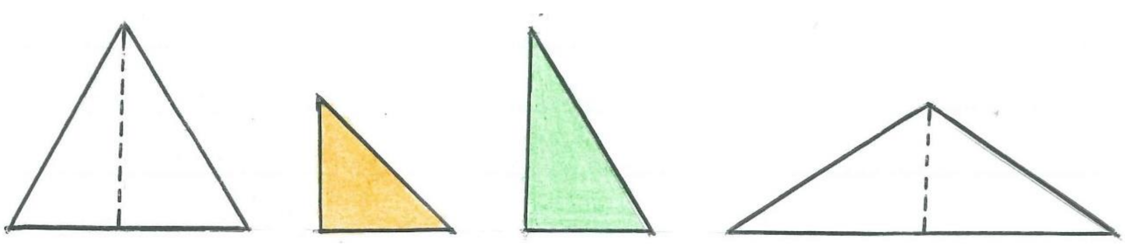
Theorem.

If a triangle has its angles $\theta_1, \theta_2, \theta_3$ all of measure a rational multiple of π, and if the sides ℓ_1, ℓ_2, ℓ_3 are in the ratio

$$\ell_1 : \ell_2 : \ell_3 = \sqrt{d_1} : \sqrt{d_2} : \sqrt{d_3},$$

where d_1, d_2, d_3 are positive rational numbers, then the triangle must be one of the four shapes with angles $(\theta_1, \theta_2, \theta_3)$ [with $0 < \theta_1 \leq \theta_2 \leq \theta_3 < \pi$] among the set

$$\{(\pi/3, \pi/3, \pi/3), (\pi/4, \pi/4, \pi/2), (\pi/6, \pi/3, \pi/2), \text{ or } (\pi/6, \pi/6, 2\pi/3)\}.$$

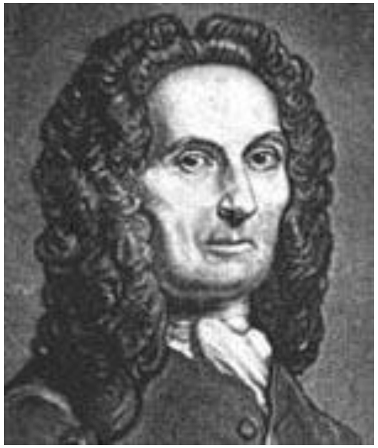


Lemma 1. If θ is a rational multiple of π, and if $\cos\theta$ is a rational number, then

$$\cos\theta \in \{0, \pm 1, \pm 1/2^k \text{ for some positive integer } k\}.$$

Lemma 2. If θ is a rational multiple of π, and if $\cos\theta$ is a rational number, then

$$\cos\theta \in \{0, \pm 1, \pm ½\}.$$



Abraham de Moivre
(1667-1754)

Lemma 2 follows from lemma 1. To prove Lemma 1 we need to express $\cos n\theta$ in terms of $\cos\theta$. One way to achieve this task is by making use of **de Moivre's Theorem**.

Multiple Angle Formula for Cosine

$e^{i\theta} = \cos\theta + i\sin\theta$, so $\cos n\theta + i\sin n\theta = e^{in\theta} = (e^{i\theta})^n$

$= (\cos\theta + i\sin\theta)^n$.

Put s = sin θ, and c = cos θ. By equating the real part,

we have

$$\cos n\theta = c^n - {}_nC_2\, c^{n-2} s^2 + {}_nC_4\, c^{n-4} s^4 - {}_nC_6\, c^{n-6} s^6 + \text{etc.}$$
$$= c^n - {}_nC_2\, c^{n-2} [1 - c^2] + {}_nC_4\, c^{n-4} [1 - c^2]^2$$
$$- {}_nC_6\, c^{n-6} [1 - c^2]^3 + \text{etc.}$$

Therefore, we obtain

$$\cos n\theta = [1 + {}_nC_2 + {}_nC_4 + \ldots + {}_nC_n](\cos\theta)^n + \ldots + (-1)^{n/2}$$
$$= 2^{n-1}(\cos\theta)^n + \ldots + (-1)^{n/2} \qquad \text{if } n \text{ is even;}$$
$$\cos n\theta = [1 + {}_nC_2 + {}_nC_4 + \ldots + {}_nC_{n-1}](\cos\theta)^n + \ldots + (0)$$
$$= 2^{n-1}(\cos\theta)^n + \ldots + (0) \qquad \text{if } n \text{ is odd.}$$

Proof of Lemma 1

Suppose $\theta = (m/n)\,\pi$ where m and n are integers, then $\cos n\theta = \cos m\pi$
$= \pm 1$.

Hence, $\cos\theta$ is a root of the equation

$$2^{n-1} x^n + \ldots + [(-1)^{n/2} \pm 1] = 0 \qquad \text{if n is even;}$$

or

$$2^{n-1} x^n + \ldots + [(0) \pm 1] = 0 \qquad \text{if n is odd .}$$

Suppose furthermore $\cos\theta = p/q$ where p and q are integers, then q is a factor of 2^{n-1} , and p is a factor of 0, ±1 or ±2. Hence, $\cos\theta$ must be of the form 0, ±1 or $\pm 1/2^k$ for some positive integer k.

Proof of Lemma 2

Note that both θ and 2θ are rational multiples of π, and that both $\cos\theta$ and $\cos 2\theta = 2(\cos\theta)^2 - 1$ are rational numbers. Hence, by Lemma 1 both $\cos\theta$ and $\cos 2\theta$ must be of the form 0, ±1 or $\pm 1/2^k$ for some positive integer k.

Since $\cos 2\theta = 2(\cos\theta)^2 - 1$, what can only happen is that k does not exceed 1, that is, $\cos\theta$ must be of the form 0, ±1 or ±1/2.

Proof of Theorem

Let us focus on each of the angles $\theta_1, \theta_2, \theta_3$ in turn, say θ_1, which for convenience will be denoted as θ. Then

$$\cos\theta = [\ell_2^{\,2} + \ell_3^{\,2} - \ell_1^{\,2}]/[2\ell_2\ell_3]$$

so that $(\cos\theta)^2$ is a rational number, because $\ell_1 : \ell_2 : \ell_3 = \sqrt{d_1} : \sqrt{d_2} : \sqrt{d_3}$, where d_1, d_2, d_3 are positive rational numbers. Hence, $\cos 2\theta = 2(\cos\theta)^2 - 1$ is also a rational number. But 2θ is also a rational multiple of π.

By Lemma 2 we see that $\cos 2\theta \in \{0, \pm 1, \pm ½\}$.

Note that $0 < 2\theta < 2\pi$ because $0 < \theta < \pi$. Hence,

$$\theta \in \{\pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}.$$

This is true for all $\theta_1, \theta_2, \theta_3$ with $\theta_1 + \theta_2 + \theta_3 = \pi$, hence we conclude that $(\theta_1, \theta_2, \theta_3)$ [with $0 < \theta_1 < \leq \theta_2 \leq \theta_3 < \pi$] is among the set

$$\{(\pi/3, \pi/3, \pi/3), (\pi/4, \pi/4, \pi/2), (\pi/6, \pi/3, \pi/2), (\pi/6, \pi/6, 2\pi/3)\}.$$

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