中國古算今譚
—— 從傳統數學至西學輸入至現代課堂數學

蕭文強
香港大學數學系
mathsiu@hku.hk

2016.05.28

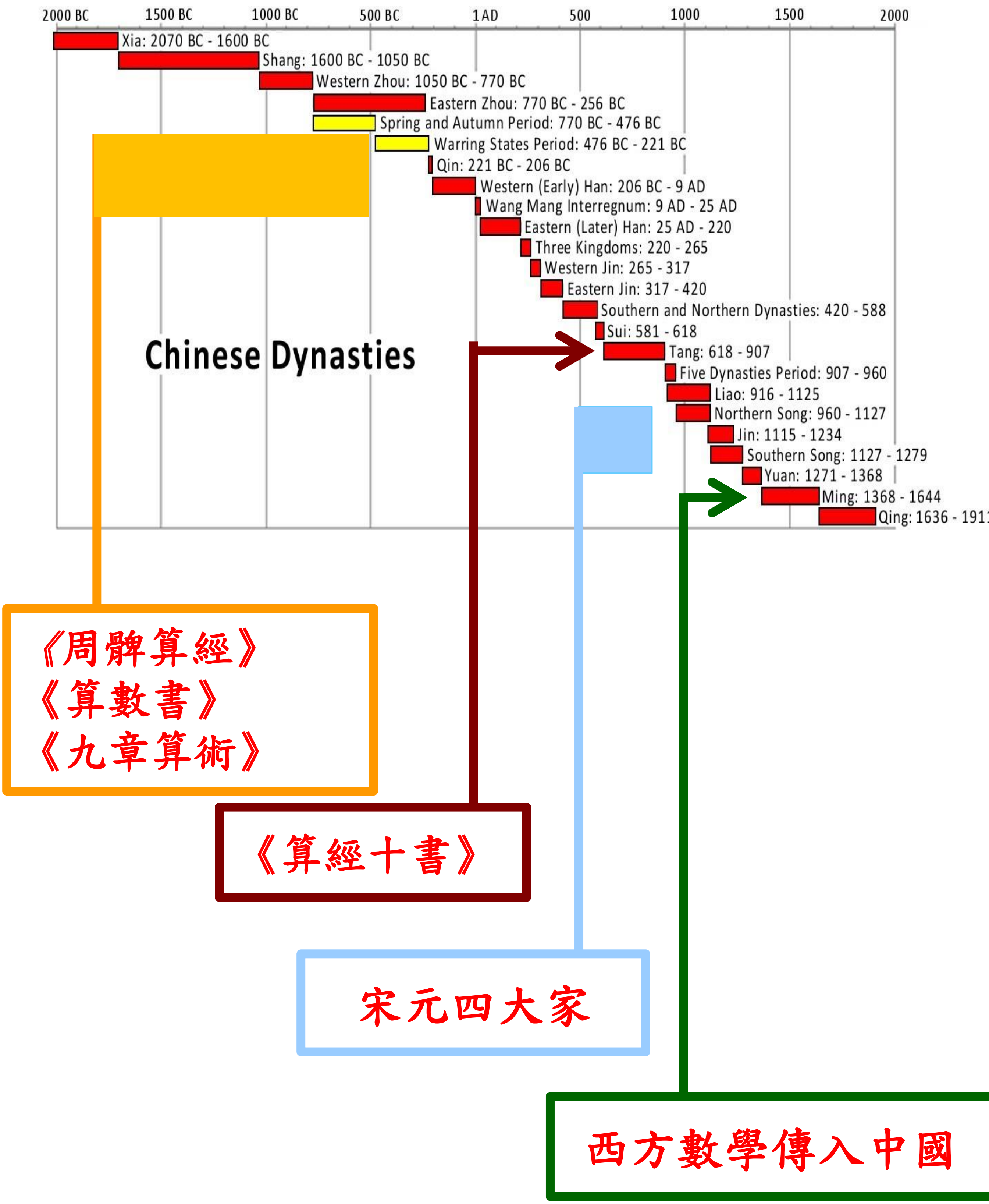
中小學數學課堂上傳授的基本知識和技能，很大部份已經有數百年以至數千年的歷史。從古代至十七世紀的東方西方數學典籍當中，記載了相當多的部份。

回顧從中國古代至十六世紀的傳統數學，至明末清初西學東漸、合流會通，演變為二十世紀以降在華人教育圈中的現代中小學數學課程內容。這方面的探討，不只有其數學意義，也富有文化意義，對教與學，都有裨益。

中小學數學課堂上傳授的基本知識和技能，很大部份已經有數百年以至數千年的歷史。從古代至十七世紀的東方西方數學典籍當中，記載了相當多的部份。

回顧從中國古代至十六世紀的傳統數學，至明末清初西學東漸、合流會通，演變為二十世紀以降在華人教育圈中的現代中小學數學課程內容。這方面的探討，不只有其數學意義，也富有文化意義，對教與學，都有裨益。

這一段故事，不可能在短短一講中作出清楚介紹，需要一系列的講座，方能敘述其中片斷。本講或可視為這項嘗試的楔子，或視為這一系列講座的「前傳」。



2000 BC
1500 BC
1000 BC
500 BC
1 AD
500
1000
1500
2000
Xia: 2070 BC - 1600 BC
Shang: 1600 BC - 1050 BC
Western Zhou: 1050 BC - 770 BC
Eastern Zhou: 770 BC - 256 BC
Spring and Autumn Period: 770 BC - 476 BC
Warring States Period: 476 BC - 221 BC
Qin: 221 BC - 206 BC
Western (Early) Han: 206 BC - 9 AD
Wang Mang Interregnum: 9 AD - 25 AD
Eastern (Later) Han: 25 AD - 220
Three Kingdoms: 220 - 265
Western Jin: 265 - 317
Eastern Jin: 317 - 420
Southern and Northern Dynasties: 420 - 588
Sui: 581 - 618
Tang: 618 - 907
Five Dynasties Period: 907 - 960
Liao: 916 - 1125
Northern Song: 960 - 1127
Jin: 1115 - 1234
Southern Song: 1127 - 1279
Yuan: 1271 - 1368
Ming: 1368 - 1644
Qing: 1636 - 1911
Chinese Dynasties
《周髀算經》
《算數書》
《九章算術》
《算經十書》
宋元四大家
西方數學傳入中國

作為中國人，我自然會滿懷自豪地學習祖先的數學成就。不過，我常常把他們的成就看作為世界整體的數學發展的一部份。猶如 David Hilbert（1862-1943）1928年在意大利Bologna 舉行的國際數學家大會(International Congress of Mathematicians）所說：

「數學無分種族。...於數學而言，整個文化世界就是單一個國家。」

M.K. Siu, *Zhi yi xing nan* (knowing is easy and doing is difficult) or vice versa? — A Chinese mathematician's observation on HPM (History and Pedagogy of Mathematics) activities, Chapter 2 in *The First Sourcebook on Asian Research in Mathematics Education: China, Korea, Singapore, Japan, Malaysia and India,* edited by B. Sriraman *et al*, Information Age Publishing, Charlotte, 2015, 27-48.

誠然，我對數學發現的先後，是東方還是西方，不感興趣。無論如何，如果為了說明本國國民的優越性，強調中國人基本上比歐洲人早了好幾個世紀發現某一數學定理，只能間接說明歐洲數學的優越性，因為這樣的比較是以西方數學為基準！

M.K. Siu, *Zhi yi xing nan* (knowing is easy and doing is difficult) or vice versa? — A Chinese mathematician's observation on HPM (History and Pedagogy of Mathematics) activities, Chapter 2 in *The First Sourcebook on Asian Research in Mathematics Education: China, Korea, Singapore, Japan, Malaysia and India,* edited by B. Sriraman *et al*, Information Age Publishing, Charlotte, 2015, 27-48.

實際上，我認
為應該以互相
學習的心態，
去察看不同的
數學文化，才
會更有成果。

M.K. Siu, *Zhi yi xing nan* (knowing is easy and doing is difficult) or vice versa? ─ A Chinese mathematician’s observation on HPM (History and Pedagogy of Mathematics) activities, Chapter 2 in *The First Sourcebook on Asian Research in Mathematics Education: China, Korea, Singapore, Japan, Malaysia and India,* edited by B. Sriraman *et al*, Information Age Publishing, Charlotte, 2015, 27-48.

中國古算(從先秦至宋元)的特色

中國古算，在內容方面，明顯具有濃厚的「經世致用」色彩。在方法上，主要著重計算(calculation)及算法(algorithms)。

M.K. Siu, An excursion in ancient Chinese mathematics, in *Using History To Teach Mathematics: An International Perspective*, edited by V. Katz, Mathematical Association of America, Washington D.C., 2000, 159-166.

中國古算(從先秦至宋元)的特色

然而，中國古算可不僅是在日常生活上應用數學的「烹飪手冊」(cookbook) 而已。

雖然它與古希臘代表作《原本》標示的作風迴異，它同樣有其結構、闡釋及證明，只不過它沒有依循古希臘的邏輯推導傳統吧。它也建立各式理論，遠超乎平凡的日常生活的需要。

M.K. Siu, An excursion in ancient Chinese mathematics, in *Using History To Teach Mathematics: An International Perspective*, edited by V. Katz, Mathematical Association of America, Washington D.C., 2000, 159-166.

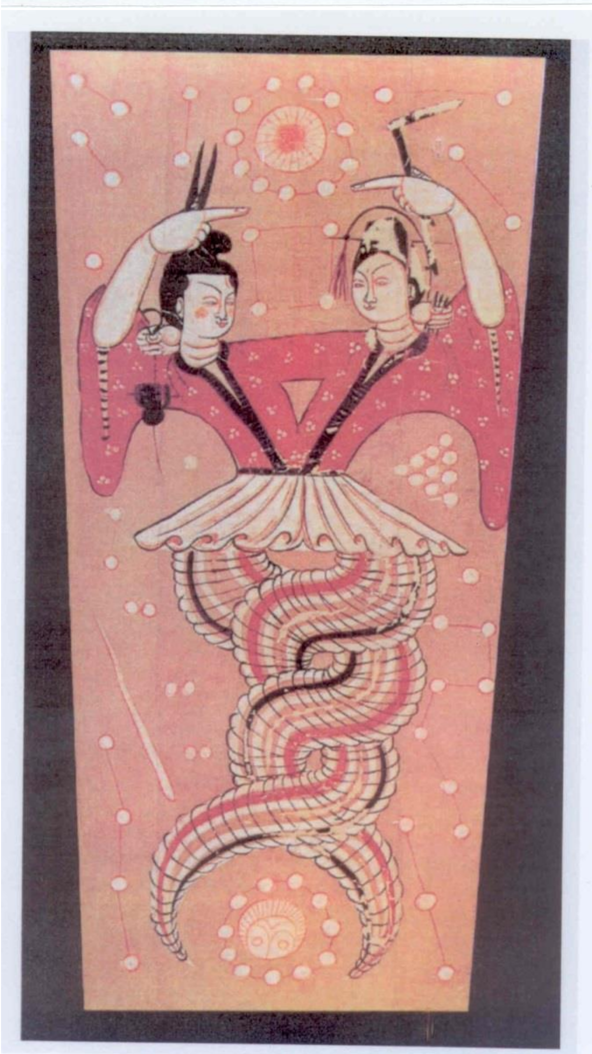
中國古算(從先秦至宋元)的特色

同時，有一個引人入勝的想法：中國古算在那個時代，不一定有如今天我們所認識的數學。在古代典籍中的確曾經出現「內算」及「外算」的提法，前者與中國最老的書本《易經》有密切關係。

M.K. Siu, An excursion in ancient Chinese mathematics, in *Using History To Teach Mathematics: An International Perspective*, edited by V. Katz, Mathematical Association of America, Washington D.C., 2000, 159-166.

漢墓石刻規矩圖:

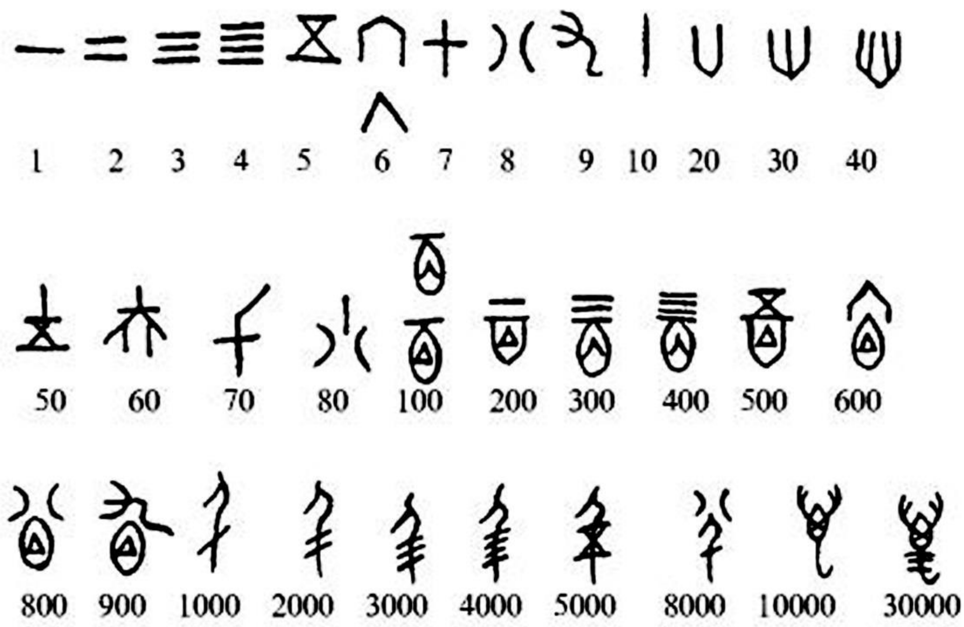
女媧執規，伏羲執矩。



彩帛規矩圖:

女媧執規，伏羲執矩。

(新疆阿斯塔那出土)



商代殷墟甲骨文的數字
(公元前十五至十一
世紀 ，河南安陽出土。)

中國自古以來已經發明，並一直沿用十進制位值制記數法。

 = 數  Number to count

 = 學 to learn, to study

to calculate

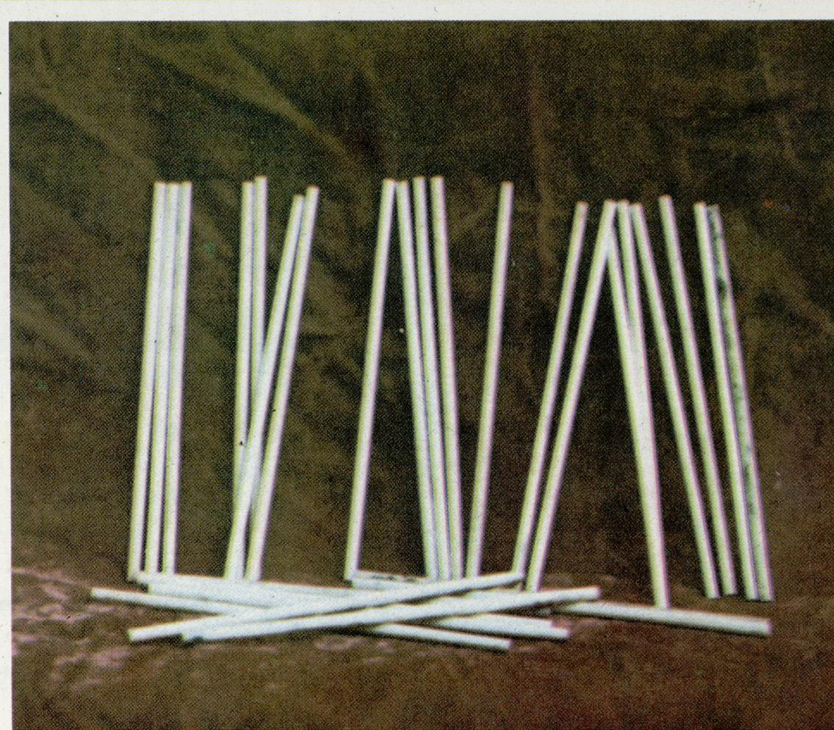
 = 算 「筭爲算之器,算爲筭之弄，二字音同而義別。」

數學(MATHEMATICS)

= 算學，算術

study of calculation, arithmetic

金属算筹（西汉）　陕西
西安东郊三店村出土
陕西省博物馆藏



象牙算筹（西汉）　陕西
旬阳出土
杜石然供稿

Counting rods　算籌

「烏賊，… 昔秦王東遊，棄算袋於海，化為此魚，形如算袋，兩帶極長。」

段成式(803–863)
《酉陽雜俎》
卷十七・廣動植之二

Squid, … Once during an excursion to the east, King Qin dropped a counting rod bag in the sea. The bag turned into this fish [squid] with a shape resembling a counting rod bag with two long strings.

Duan Chengshi, Book 17, *Youyang Zazu*
(9th century)

玉方寸重一十二兩　銅方寸重七兩半
鈆方寸重九兩半　鐵方寸重六兩
石方寸重三兩
凡筭之法先識其位一從十橫百立千僵千十
相望百萬相當
凡乘之法重置其位上下相觀上位有十步至
十有百步至百有千步至千以上命下所得之
數列於中位言十即過不滿自如上位乘訖者
先去之下位乘訖者則俱退之六不積五不隻

Sunzi Suanjing **(孫子算經 [Master Sun's Mathematical Manual]), 4th/5th century.**

縱式

$1, 10^2, 10^4, \ldots$

夏侯陽算經

卷上

一

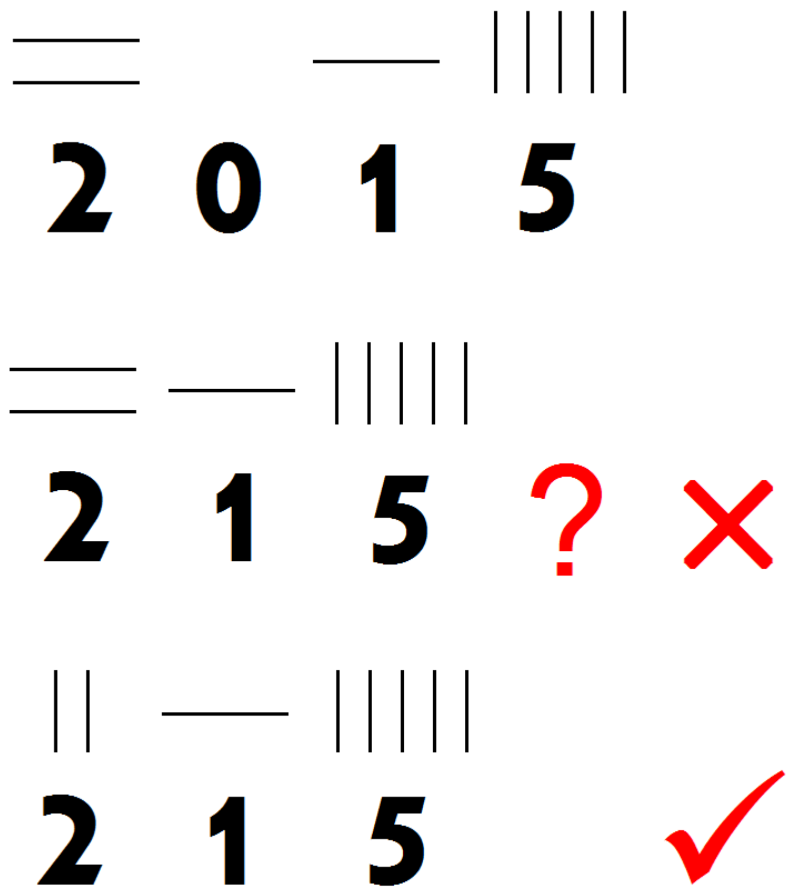
項家達校

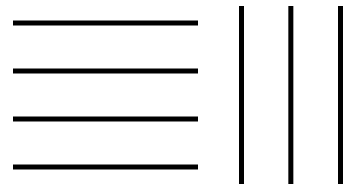
之銖之求絫黍皆上十之斗之求升合抄撮皆上十之
里之求步三百之步之求尺六之氂毫絲忽可以意卸
夫乘除之法先明九九一從十横百立千僵千十相望
萬百相當滿六已上五在上方六不積算五不單張上
下相乘實居中央言十自過不滿自當以法除之宜得
上商從算相似横算相當以次右行極于左方言法之
上見十步至十見百步至百見千步至千見萬步至萬

横式

$10, 10^3, \ldots$

夏侯陽算經（公元五世紀）


2 0 1 5
2 1 5 ?
2 1 5

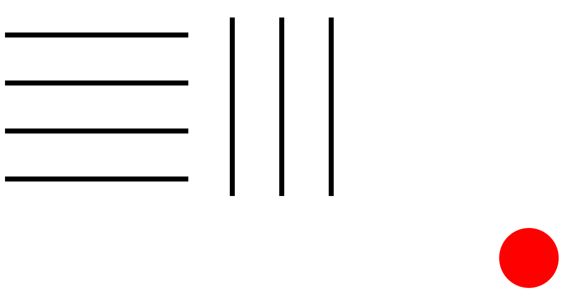


4 3

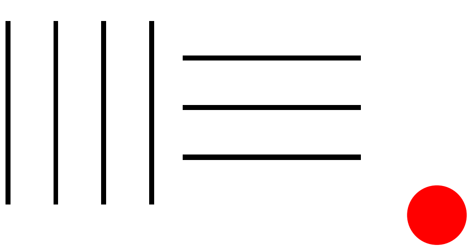
Can it mean 4 3 0 0 ?

「凡筭者，正身端坐，一從右膝而起。」(All calculators [who manipulate counting rods] sit upright with the right knee signifying the unit position.)

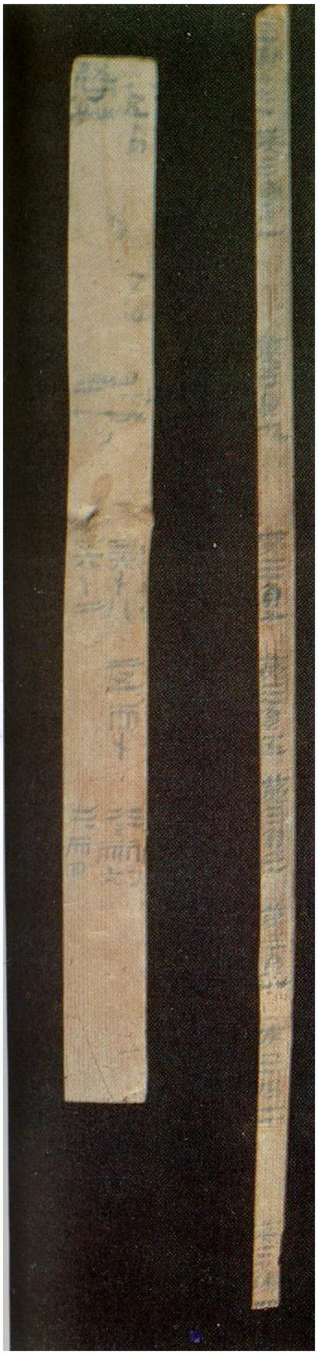
《敦煌算書》[Dunhuang Manuscript on Arithmetical Calculation], 4th– 6th centuries



= 4 3 0 0



= 4 3 0



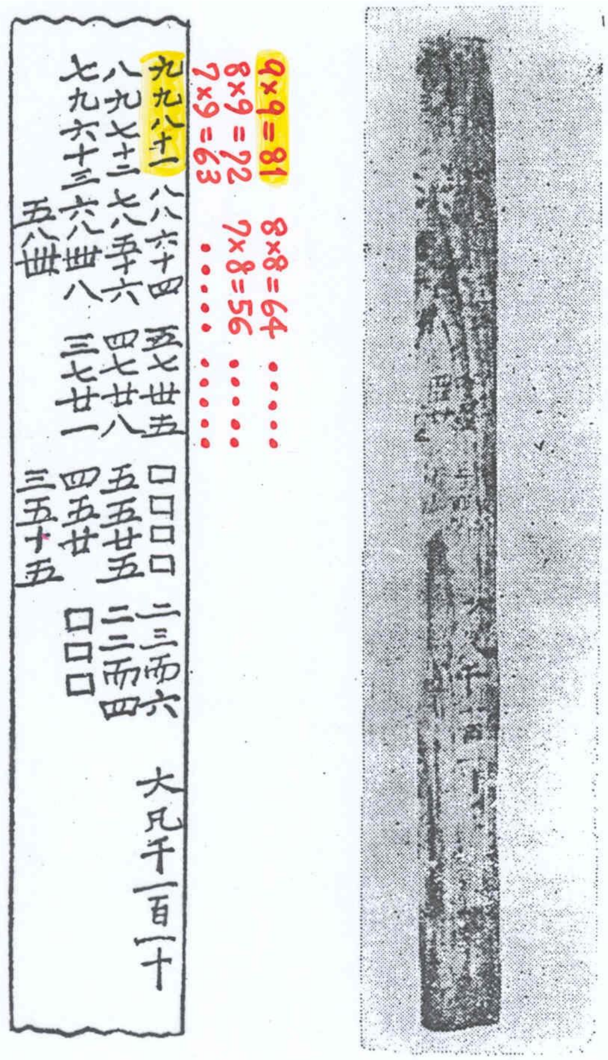
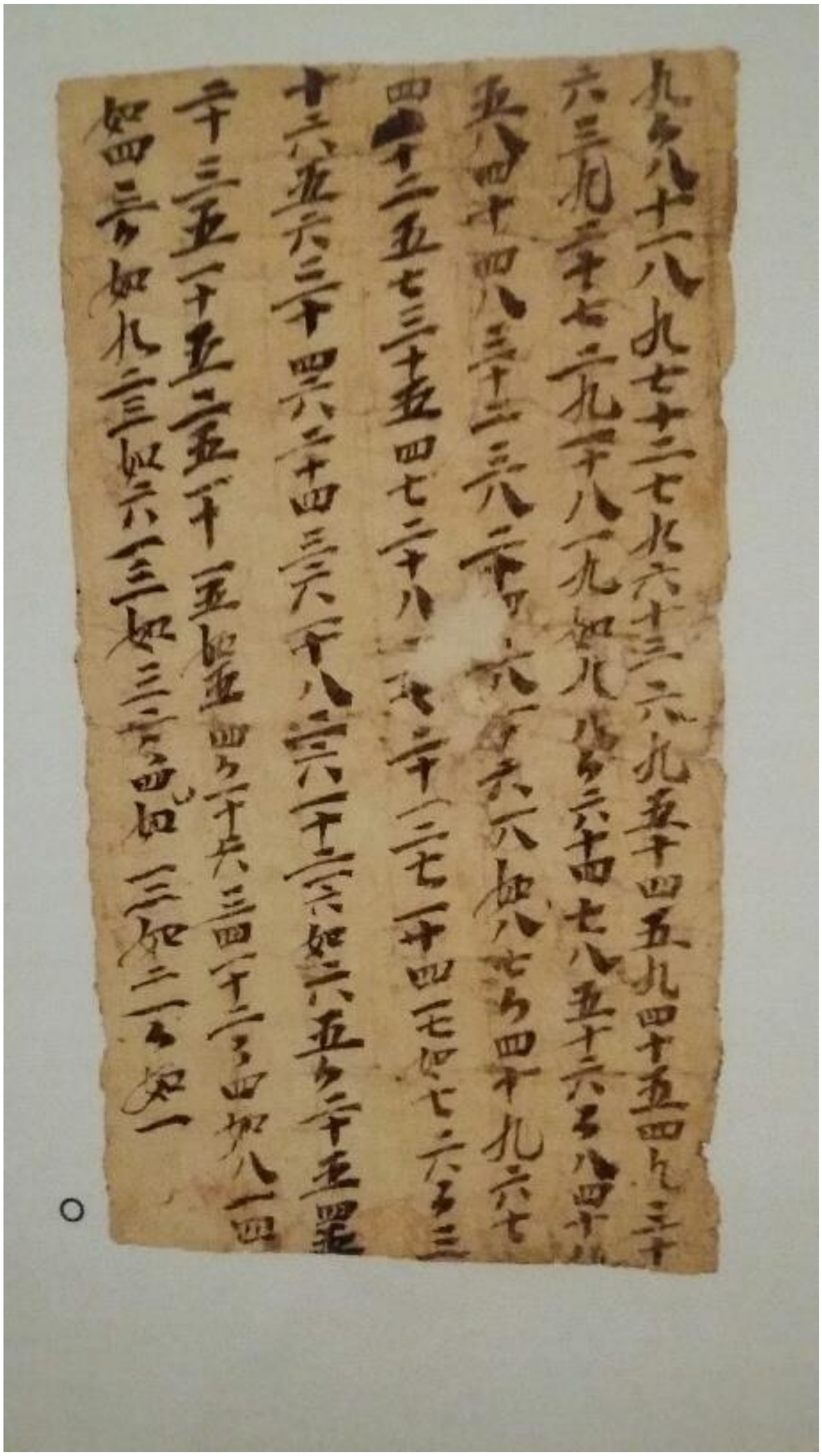


圖 2　漢簡九九圖
（採自《流沙墜簡》）

漢簡九九歌



敦煌殘卷中的算書上的**九九表**
(公元四至六世紀)

2	5	4	6
	3	7	1

(1)

2	8	4	6
		7	1

(2)

2	9	1	6
			1

(3)

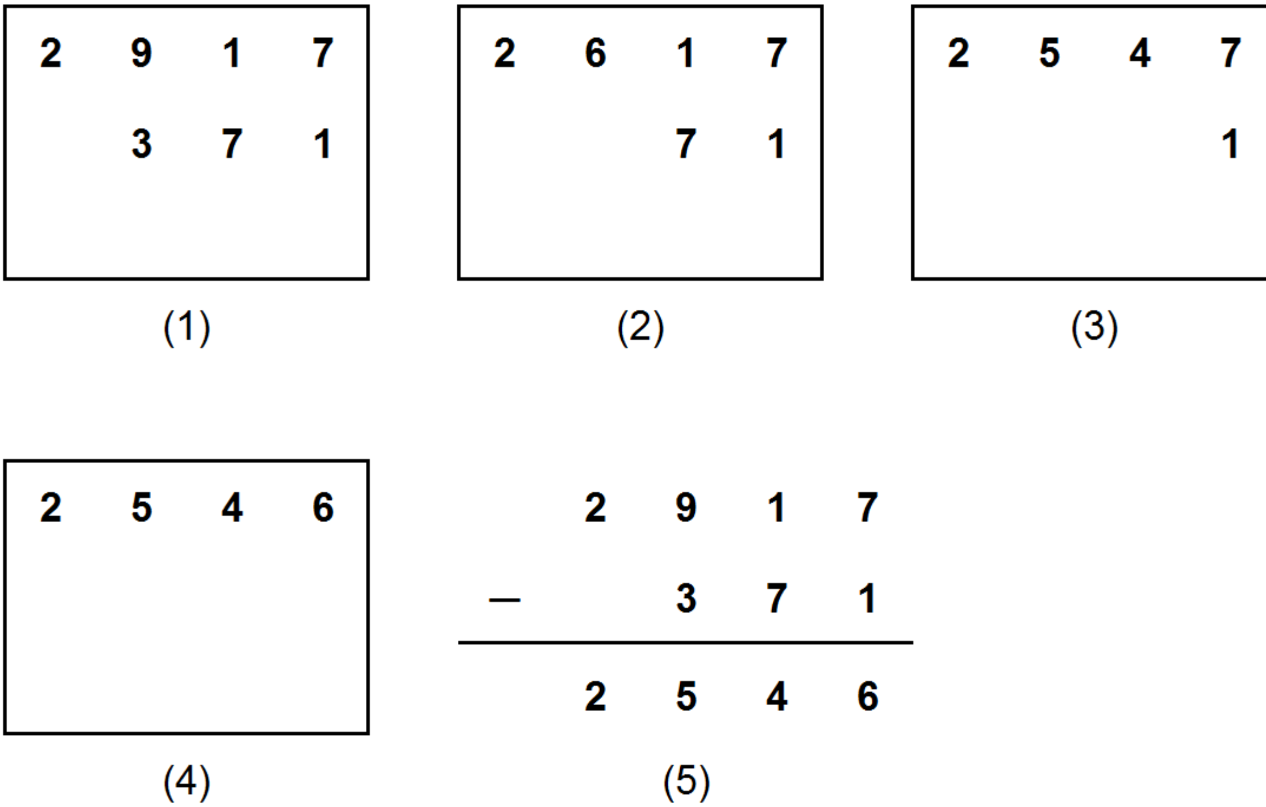
2	9	1	7

(4)

$$\begin{array}{ccccc} & 2 & 5 & 4 & 6 \\ + & & 3 & 7 & 1 \\ \hline & 2 & 9 & 1 & 7 \end{array}$$

(5)

Addition [implicitly explained in *Sunzi Suanjing* (孫子算經 [Master Sun's Mathematical Manual]), 4th/5th century.]



Subtraction [implicitly explained in *Sunzi Suanjing* (孫子算經 [Master Sun's Mathematical Manual]), 4th/5th century.]

	7	2	3	9
2	3			

		7	2	3	9
1	4				
	2	3			

		7	2	3	9
1	6	1			
	2	3			

			2	3	9
1	6	1			
		2	3		

			2	3	9
1	6	5	6		
		2	3		

				3	9
1	6	5	6		
			2	3	

				3	9
1	6	6	2	9	
			2	3	

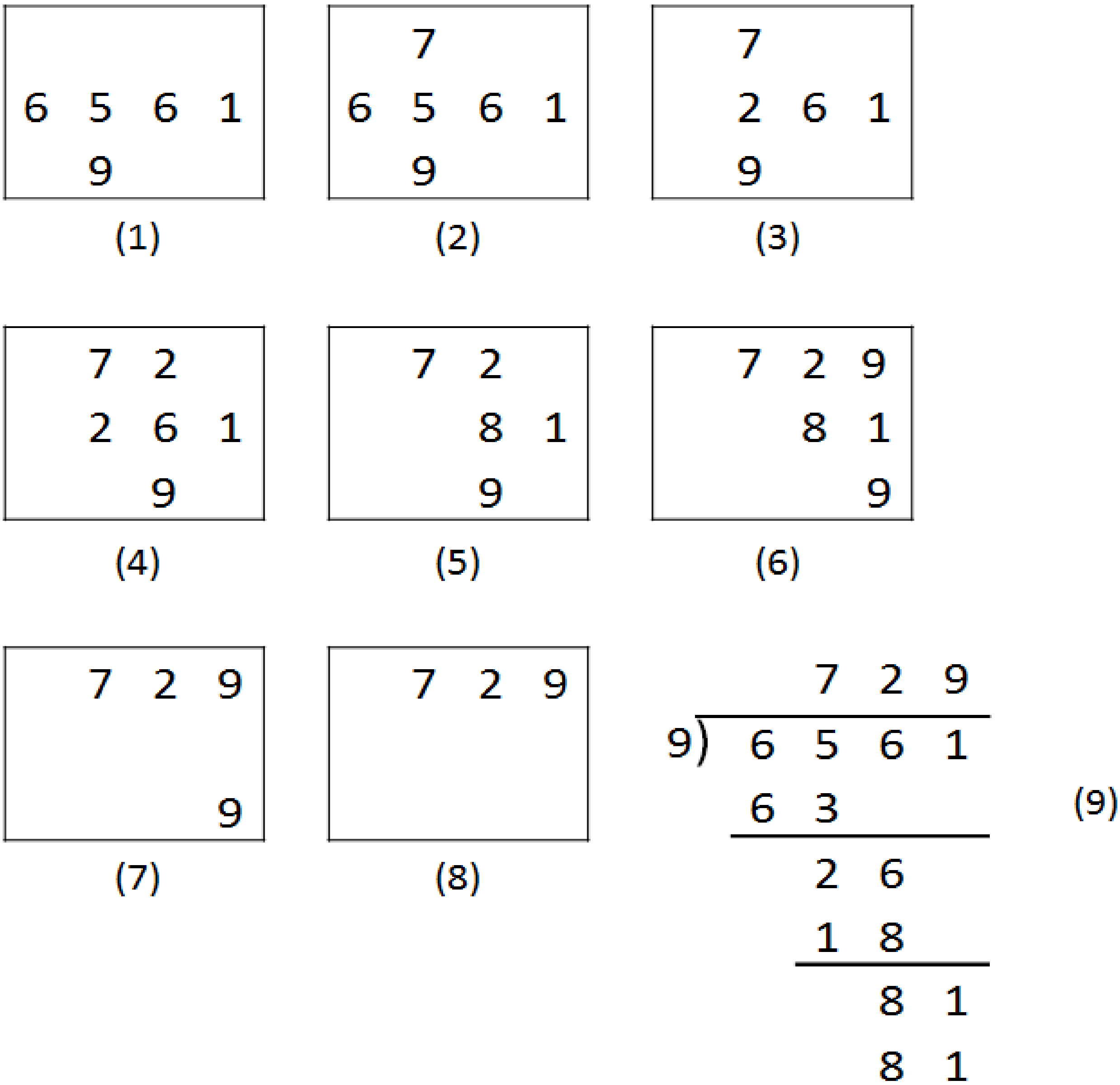
					9
1	6	6	2	9	
				2	3

					9
1	6	6	4	9	7
				2	3

1	6	6	4	9	7

		7	2	3	9
x				2	3
1	4				
	2	1			
		4			
			6		
			6		
				9	
			1	8	
				2	7
1	6	6	4	9	7

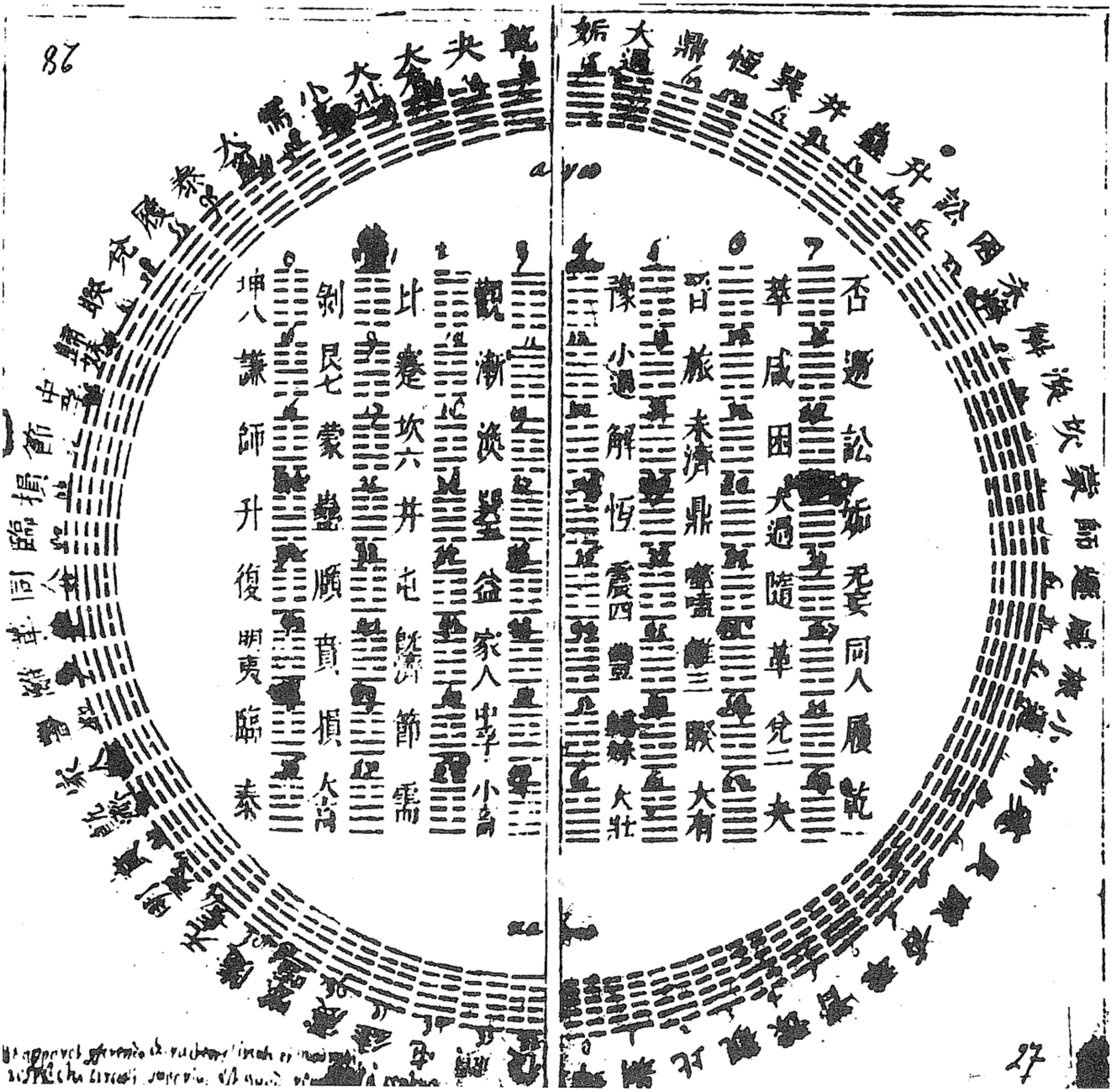
Multiplication explained in *Sunzi Suanjing* (孫子算經 [Master Sun's mathematical manual]), 4th/5th century



Division explained in *Sunzi Suanjing* (孫子算經 [Master Sun's Mathematical Manual]), $4^{th}/5^{th}$ century.

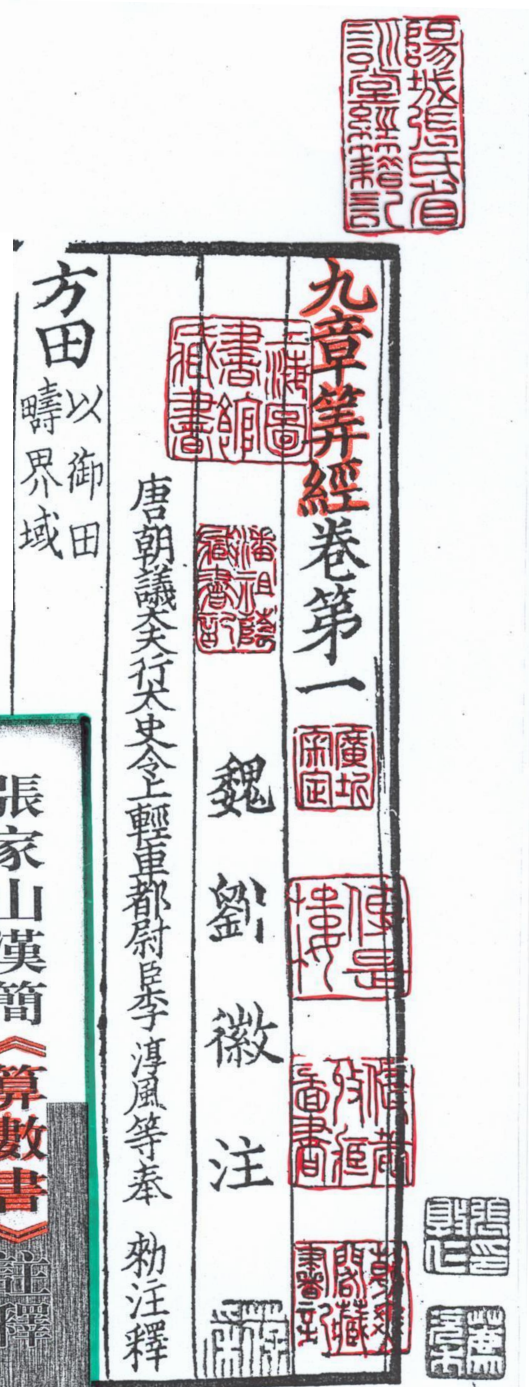
Hong Kong Definitive Stamp (Issue on October 14, 2002)

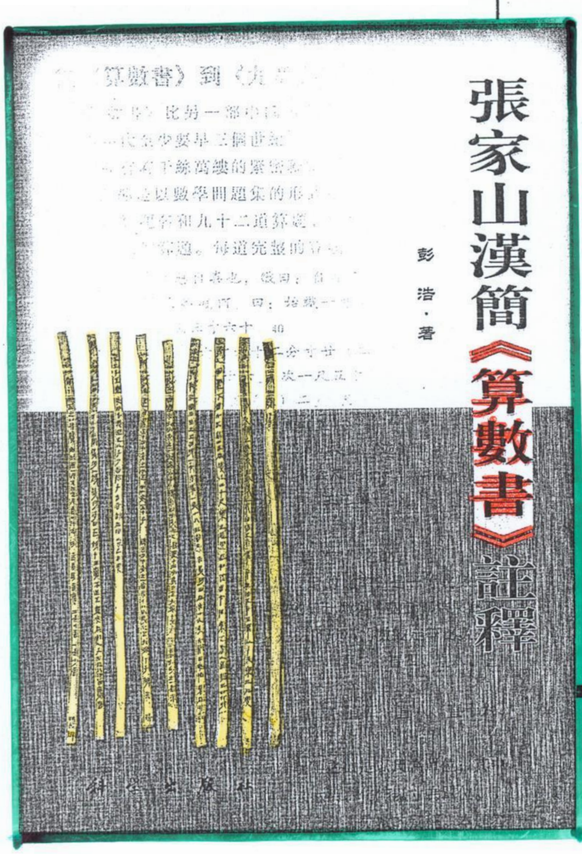
Calculator/Abacus



A diagram of the *A Priori (Natural) Hexagram Order* (先天六十四卦次序) in *Yijing* (易經 Book of Changes) sent by the Jesuit Father Joachim Bouvet to Gottfried Wilhelm Leibniz in 1701. Bouvet was struck by the similarity this bears to the binary system of arithmetic developed by Leibniz.

Jiuzhang Suanshu
《九章算術》
(Nine Chapters on
the Mathematical Art)
compiled between
100 B.C.E. to 100 C.E.





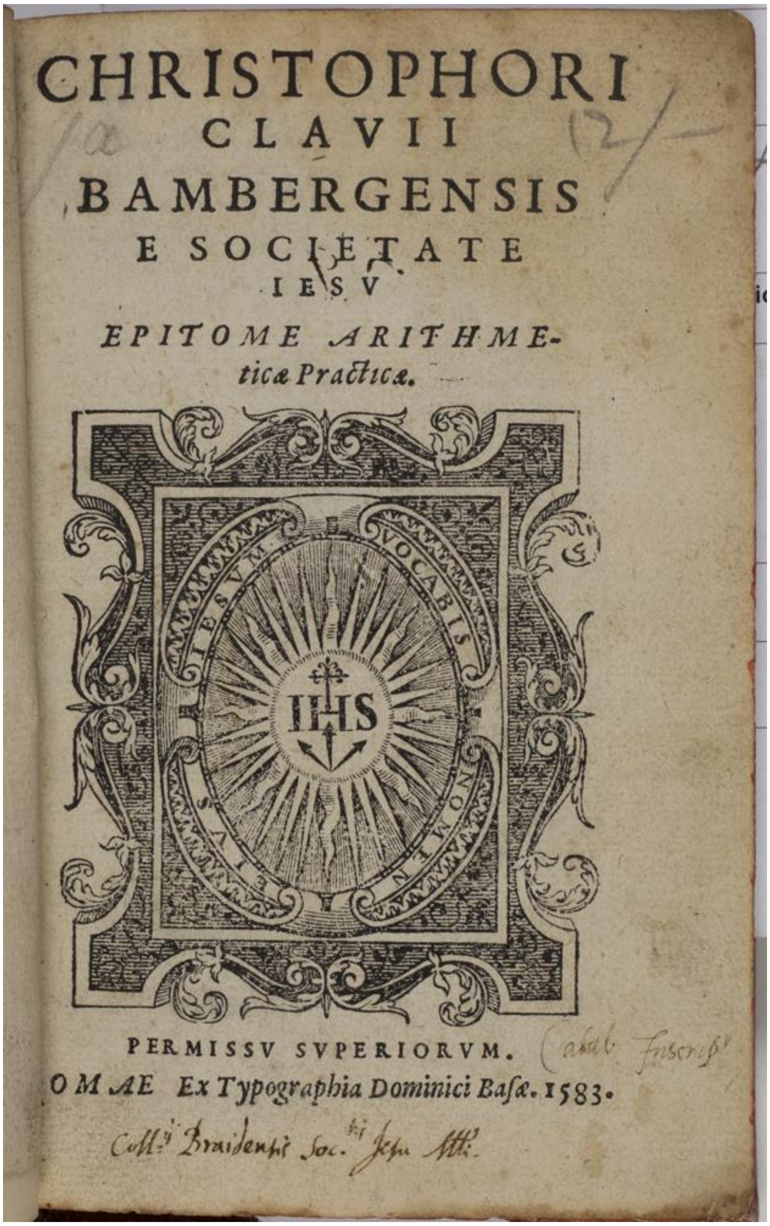
Suanshu Shu 《算數書》
(Book of Numbers and
Computation)
ca. 200 B.C.E.

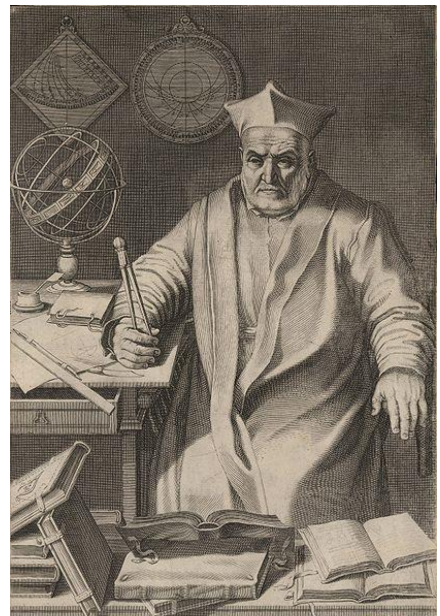
Suanshu Shu **《算數書》(Book of Numbers and Computation), *ca*. 200 B.C.E.**
Excavated in Zhangjiashan in Hubei Province in 1983

《九章算術》***Jiuzhang Suanshu*** **(Nine Chapters on the Mathematical Art), compiled between 1st century B.C.E. and 1st century C.E., with 246 problems grouped into nine chapters.**

目錄 附

CHENG Da-wei (程大位）
Suanfa Tongzong
(算法統宗, literally meaning “unified source of computational methods”) , 1592.





Christopher Clavius (1538-1612)

Epitome Arithmeticae Practicae **[literally meaning "abridgement of arithmetic in practice"] compiled by Christopher CLAVIUS, 1583.**

Tongwen Suanzhi **[同文算指, literally meaning "rules of arithmetic common to cultures"] compiled by Li Zhi-zao (李之藻) and Matteo RICCI (利瑪竇), 1613.**

數之原其與生人俱來乎？始於一，終於十，十指象之，屈而計諸，不可勝用也。
五方萬國，風習千變，至於算數，無弗同者，十指之賅存，無弗同耳。

徐光啟•刻《同文算指》序
(1613)

東海西海，心同理同。所不同者，特言語文字之際。

[Across the seas of the East and the West the mind and reasoning are the same. The difference lies only in the language and the writing.]

Li Zhi-zhao (李之藻), Preface to the reprinting of *Tianzhu Shiyi* [天主實義重刻序]

Tianzhu Shiyi **[The True Meaning of the Lord of Heaven 天主實義] was written by Matteo Ricci (利瑪竇) and printed in 1603 in Peking.**

寓數於形，表形以數。
數形結合，雙翼齊飛。

「算術及幾何，天文學家藉著這對翅膀翱翔天際，與天比高。」

Robert Boyle
(1627-1691)

動態幾何軟件 *GeoGebra* 集算術及幾何之長，正好配合中國古算的優點，從中擷取合適的歷史素材，用以製作程序顯示，以輔助教學。

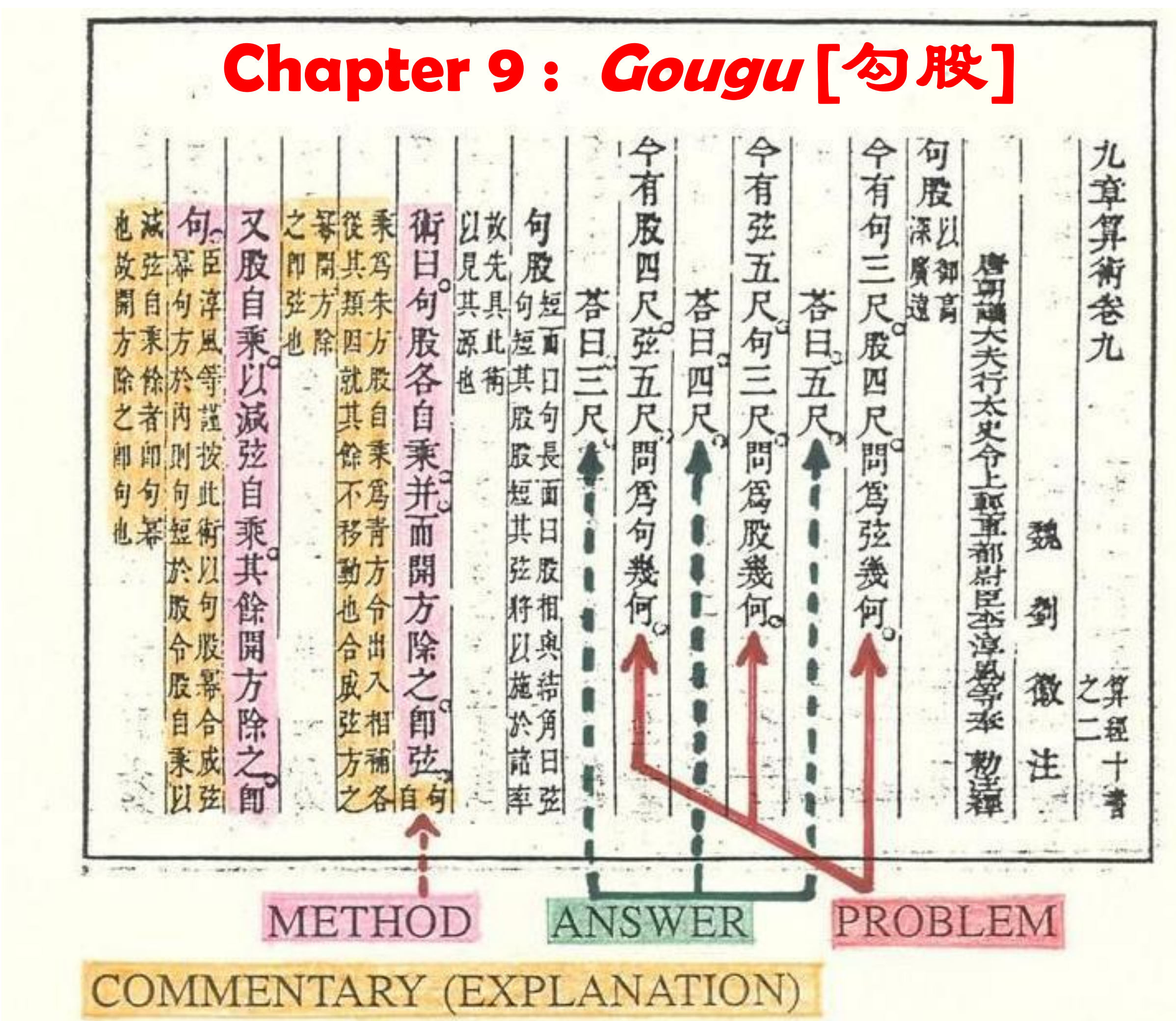
在以下的時段，我們從中國古代數學典籍中擷取不同的例子，以展示上述這一點，視乎餘下的時間有多少，便講多少。

問題選自：

- 《九章算術》
- 《海島算經》
- 《孫子算經》

Jiuzhang Suanshu [九章算術 Nine Chapters on the Mathematical Art]
_ca._ 100 B.C.E. — 100 C.E.

Commentary on _Jiiuzhang Suanshu_ by LIU Hui [劉徽]



GUO Shuchun (郭書春), _Jiuzhang Suanshu Yizhu_ [九章算術譯注 Translation and Annotation of the Nine Chapters on Mathematical Procedures], 2009.

- **Variation Theory** in teaching/learning (Ference Marton and his team)

- ***Bianshi* method** [變式] in teaching/learning (Gu Ling-yuan [顧泠沅] and his team)

問題1： 今有勾三尺，股四尺，問：為弦幾何？

問題2： 今有弦五尺，勾三尺，問：為股幾何？

問題3： 今有股四尺，弦五尺，問：為勾幾何？

變式的「最低級版」

問題 4： 今有圓材徑二尺五寸，欲為方版，令厚七寸。問：廣幾何？

問題 5： 今有木長二丈，圍之三尺。葛生其下，纏木七周，上與木齊。問：葛長幾何？

昇了一級的「故事題」

問題 6： 今有池方一丈，葭生其中央，出水一尺。引葭赴岸，適與岸齊。

問：水深、葭長各幾何？

問題 7： 今有立木，繫索其末，委地三尺。引索卻行，去本八尺而索盡。

問：索長幾何？

變式的「高級版」

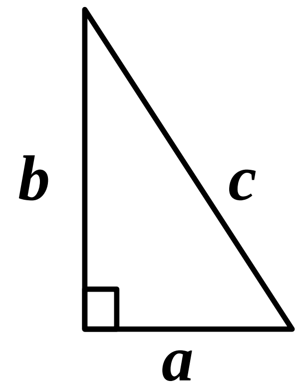
問題 11：今有戶高多於廣六尺八寸，兩隅相去適一丈。問：戶高、廣各幾何？

問題 12：今有戶不知高、廣，竿不知長短。橫之不出四尺，從之不出二尺，邪之適出。問：戶高、廣、袤各幾何？

●●●●●●●●●

Given two out of the nine quantities
a, b, c, $a + b$, $b + c$, $a + c$, $b - a$,
$c - a$, $c - b$, calculate the remaining ones.

(The problems refer to those in
Chapter 9 of *Jiuzhang Suanshu*.)



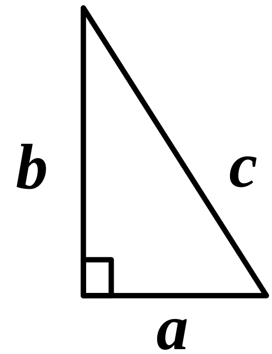
$c > b > a$

❶	a, b		Problem 1, 5.
❷	a, c		Problem 2, 4.
	a, $a + b$	❶	
❸	a, $b + c$		Problem 13.
	a, $a + c$	❷	Problem 14.
	a, $b - a$	❶	
	a, $c - a$	❷	
❹	a, $c - b$		Problem 6, 7, 8, 9, 10.

	b, c	❷	**Problem 3.**
	b, $a + b$	❶	
	b, $b + c$	❷	
	b, $a + c$	❸	
	b, $b - a$	❶	
	b, $c - a$	❹	
	b, $c - b$	❷	
❺	c, $a + b$		
	c, $b + c$	❷	
	c, $a + c$	❷	
❻	c, $b - a$		**Problem 11.**
	c, $c - a$	❷	
	c, $c - b$	❷	

❼	$a+b,\quad b+c$		
	$a+b,\quad a+c$	❼	
	$a+b,\quad b-a$	❶	
	$a+b,\quad c-a$	❼	
	$a+b,\quad c-b$	❼	
❽	$b+c,\quad a+c$		
	$b+c,\quad b-a$	❽	
	$b+c,\quad c-a$	❼	
	$b+c,\quad c-b$	❷	
	$a+c,\quad b-a$	❽	
	$a+c,\quad c-a$	❷	
	$a+c,\quad c-b$	❼	
❾	$b-a,\quad c-a$		
	$b-a,\quad c-b$	❾	
	$c-a,\quad c-b$	❾	**Problem 12.**

Given two out of the nine quantities
***a*, *b*, *c*, *a* + *b*, *b* + *c*, *a* + *c* , *b* – *a*,**
***c* – *a*, *c* – *b*, calculate the remaining ones.**



$c > b > a$

❶	a,	b	*Jiuzhang Suanshu*
❷	a,	c	*Jiuzhang Suanshu*
❸	a,	$b + c$	*Jiuzhang Suanshu*
❹	a	$c - b$	*Jiuzhang Suanshu*
❺	c,	$a + b$	Zhao Shuang (3rd century)
❻	c,	$b - a$	*Jiuzhang Suanshu*
❼	$a + b$,	$b + c$	Xiang Mingda (1825)
❽	$b + c$,	$a + c$	Zhu Shijie (1299)
❾	$c - a$,	$c - b$	*Jiuzhang Suanshu*

勾和和	$(c+b)+a$
勾較較	$a-(c-b)$
勾較和	$(c-b)+a$
勾和較	$(c+b)-a$

Together with a, b, c, $a + b$, $b + c$, $a + c$, $b - a$, $c - a$, $c - b$, there are 13 quantities, with 78 combinations two chosen at a time. LI Rui reduced these to basically 25 types and treated all of them.

李鋭•《勾股算術細草》(1806)

❖ Problems 1 to 14

Solution of a right triangle

(*In-Out Principle* — dissecting and re-assembling suitable pieces)

❖ Problems 15 and 16 *

Inscribed square or circle in a right triangle

(In-Out Principle / *Bilü* [rates] between sides retained)

❖ Problems 17 to 24 *

Surveying problems involving a right triangle

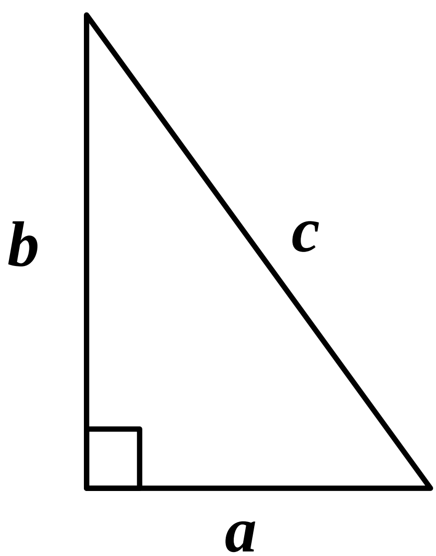
(application of *Bilü* [rates])

* Problems 14 and 21 actually deal with Pythagorean triplets.

Recruitment advertisement in a magazine in the Netherlands, April 27, 1999

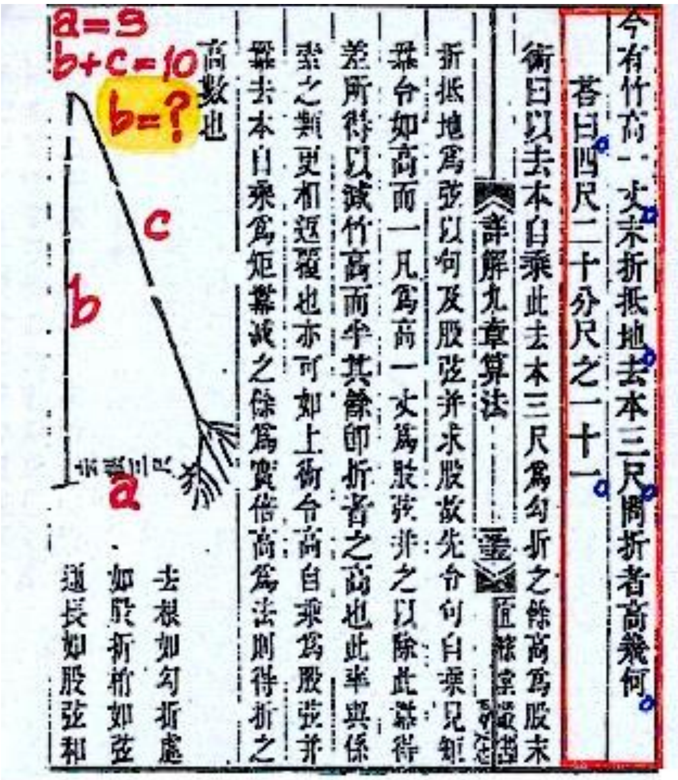
Recruitment advertisement in a magazine in the Netherlands, April 27, 1999

Now given a bamboo 1 *zhang* high, which is broken so that its tip touches the ground 3 *chi* away from the base.
Tell: what is the height of the break?



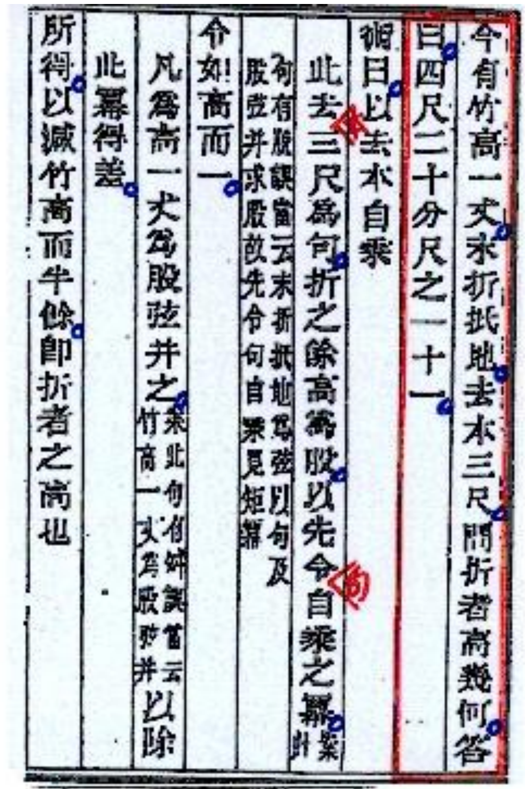
Jiuzhang Suanshu
Chapter 9 ,
Problem 13.

The problem also appeared (with different data) in Bhāskara's *Lilavati* (12th century) and Calandri's text (15th century).



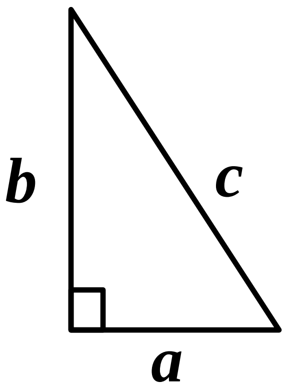
今有竹高一丈末折抵地去本三尺問折者高幾何
荅曰四尺二十分尺之一十一
術曰以去本自乘此去本三尺為勾折之餘高為股末
《詳解九章算法》
折抵地為弦以句及股弦并求股故先令句自乘見矩
冪令如高而一凡為高一丈為股弦并之以除此冪得
差所得以減竹高而半其餘即折者之高也此率與係
索之類更相返覆也亦可如上術令高自乘為股弦并
冪去本自乘為矩冪減之餘為實倍高為法則得折之
高數也

YANG Hui (楊輝), ***A Detailed Analysis of the Mathematical Methods in the 'Nine Chapters'***
《詳解九章算法》(1261)


今有竹高一丈末折抵地去本三尺問折者高幾何答
曰四尺二十分尺之一十一
術曰以去本自乘
此去三尺為句折之餘高為股以先令句自乘之冪
句有股誤當云末折抵地為弦以句及
股弦并求股故先令句自乘見矩冪
令如高而一
凡為高一丈為股弦并之
案此句有舛誤當云
竹高一丈為股弦并以除
此冪得差
所得以減竹高而半餘即折者之高也

Jiuzhang Suanshu
(Nine Chapters on the Mathematical Art)
《九章算術》
(ca.100 B.C.E.- 100 C.E.)

$$b = \frac{1}{2}\left[(c+b) - \frac{a^2}{(c+b)}\right]$$



Given a and $c + b$, calculate b.

Solution by a school pupil of today

Let $L = c + b$, then $c = L - b$.

But $a^2 + b^2 = c^2 = (L - b)^2$

$$= L^2 - 2Lb + b^2,$$

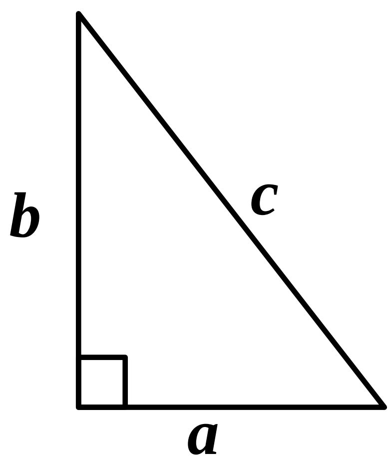
so $\quad a^2 = L^2 - 2Lb,$

or $\quad 2Lb = L^2 - a^2\,,$

$$b = \frac{1}{2}\left[L - \frac{a^2}{L}\right],$$

$$\boxed{b = \frac{1}{2}\left[(c+b) - \frac{a^2}{(c+b)}\right].}$$

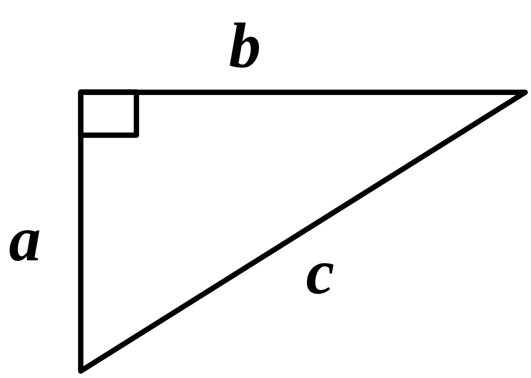
How was it done at the time, which was more than two thousand years ago?



Given a and $c + b$, calculate b.

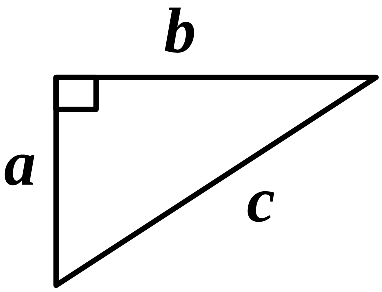
http://ggbtu.be/m2772025

Two persons *A* (*Jia*) and *B* (*Yi*) stood at the same spot. In the time when *A* walked 7 steps, *B* could walk 3 steps. *B* walked east and *A* walked south. After 10 steps south *A* turned to walk in a roughly northeast direction to meet *B*. How many steps had each walked (when they met)?



Given $(a + c) : b = 7 : 3$ and $a = 10$, calculate b and c.

Jiuzhang Suanshu
Chapter 9 , Problem 14.



Given $(a + c) : b = 7 : 3$ and $a = 10$, calculate b and c.

Solution by a school pupil of today

$a = 10,\ 3(a + c) = 7b$ and $c^2 = a^2 + b^2$.

Therefore $$c^2 = a^2 + \frac{9}{49}(a + c)^2$$
$$= 100 + \frac{9}{49}(10 + c)^2 ,$$

or $49\,c^2 = 4900 + 900 + 180\,c + 9\,c^2$,

$40\,c^2 = 5800 + 180\,c$,

$2\,c^2 - 9\,c - 290 = 0.$

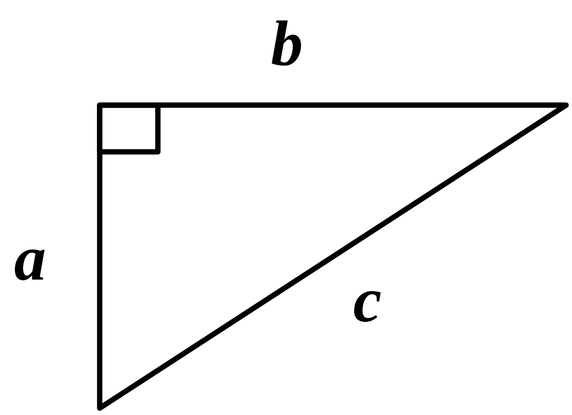
Hence, $c = 14\frac{1}{2}$ or -10 (which is an inadmissible root) .

So $c = 14\frac{1}{2}$ and $b = \frac{3}{7}(10 + c) = 10\frac{1}{2}$.

The solution actually yields a method to construct **Pythagorean triplets!**

$$a : b : c$$
$$= (m^2 - n^2)/2 : mn : (m^2 + n^2)/2$$

where $(a + c) : b = m : n$.



http://ggbtu.be/m2754837

❖ Problems 1 to 14

Solution of a right triangle

(*In-Out Principle* – dissecting and re-assembling suitable pieces)

❖ Problems 15 and 16

Inscribed square or circle in a right triangle

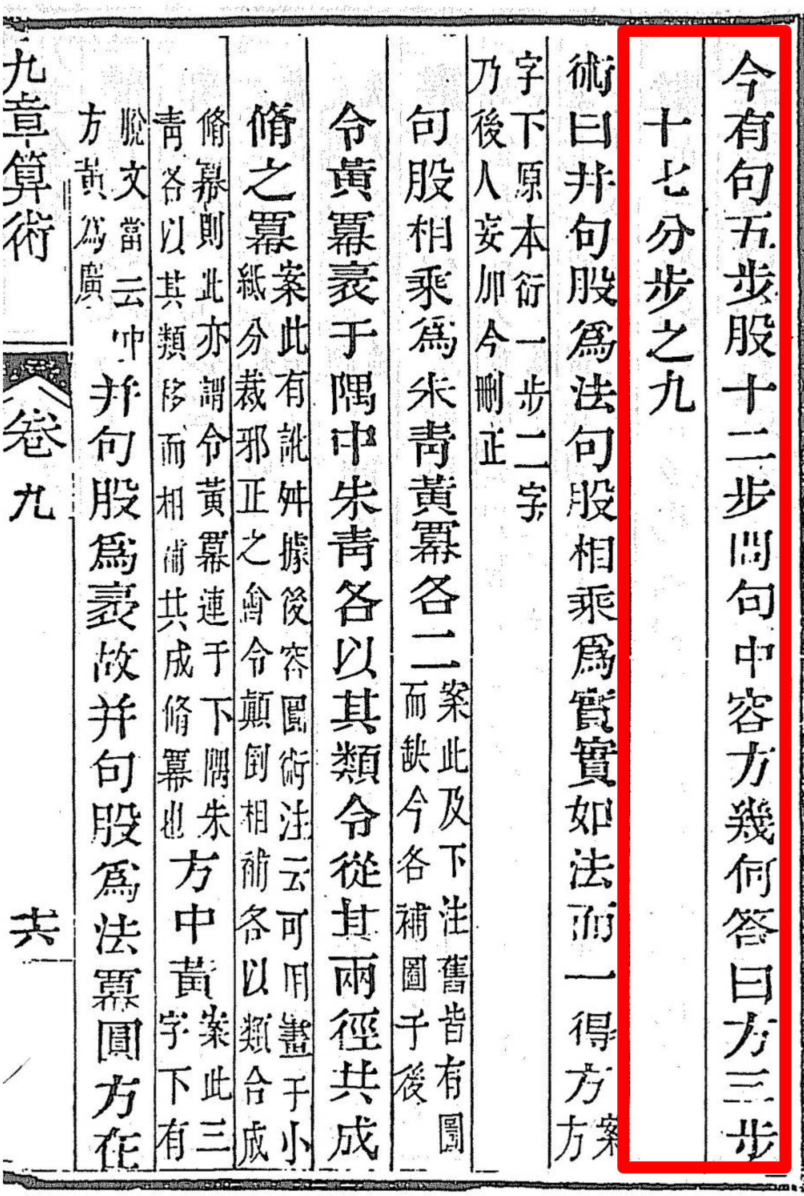
(In-Out Principle / *Bilü* [rates] between sides retained)

❖ Problems 17 to 24

Surveying problems involving a right triangle

(application of ***Bilü* [rates]**)

LI Jimin (李繼閔), 《九章算術》及其劉徽注研究 [Study on *Jiuzhang Suanshu* and its Commentary by Liu Hui],1990.



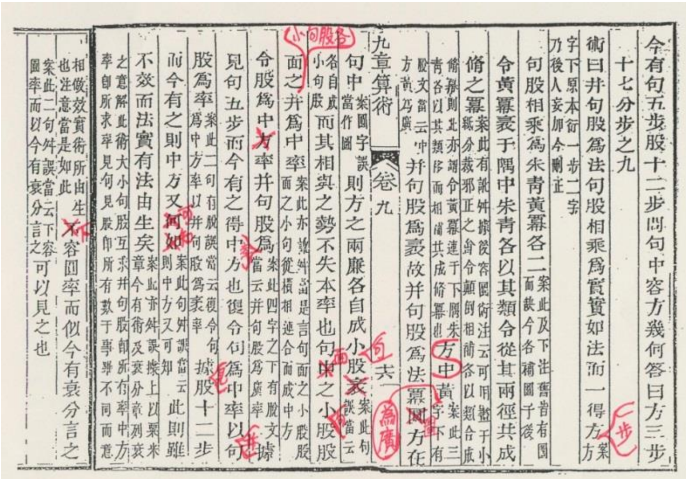
今有句五步股十二步問句中容方幾何答曰方三步
十七分步之九
術曰并句股爲法句股相乘爲實實如法而一得方
字下原本衍一步二字乃後人妄加今刪正
句股相乘爲朱青黃冪各二 案此及下注舊皆有圖而缺今各補圖于後
令黃冪袤于隅中朱青各以其類令從其兩徑共成
脩之冪 案此有訛舛據後容圓術注云可用畫于小紙分裁邪正之會令顛倒相補各以類合成
脩冪則此亦謂令黃冪連于下隅朱青各以其類移而相補共成脩冪也
方中黃 案此三字下有脫文當云中方黃爲廣
并句股爲袤故并句股爲法冪圓方在

九章算術 卷九 十六

***Jiuzhang Suanshu* [九章算術, compiled between 100 B.C.E. and 100 C.E.] Chapter 9, Problem 15.**

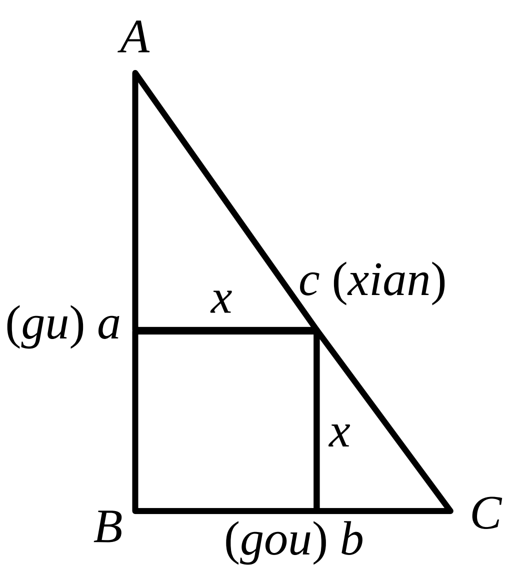
Now given a right-angled triangle whose *gou* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*.

Jiuzhang Suanshu [九章算術，compiled between 100 B.C.E. and 100 C.E.] , Chapter 9, Problem 15.


今有句五步股十二步問句中容方幾何答曰方三步十七分步之九
術曰并句股為法句股相乘為實實如法而一得方步

Now given a right-angled triangle whose *gou* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*.

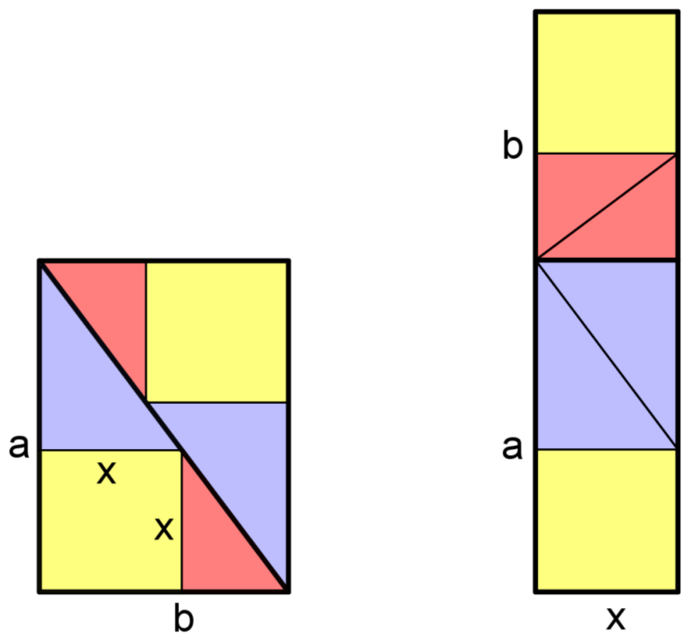
Method: Let the sum of the *gou* and the *gu* be the divisor; let the product of the *gou* and the *gu* be the dividend. Divide to obtain the side of the square.



$$\boxed{x = \frac{ab}{a+b}}$$

Commentary by LIU Hui (劉徽) [mid 3rd century]

Method 1 (dissect-and-re-assemble)



Area = ab Area = $(a + b)\,x$

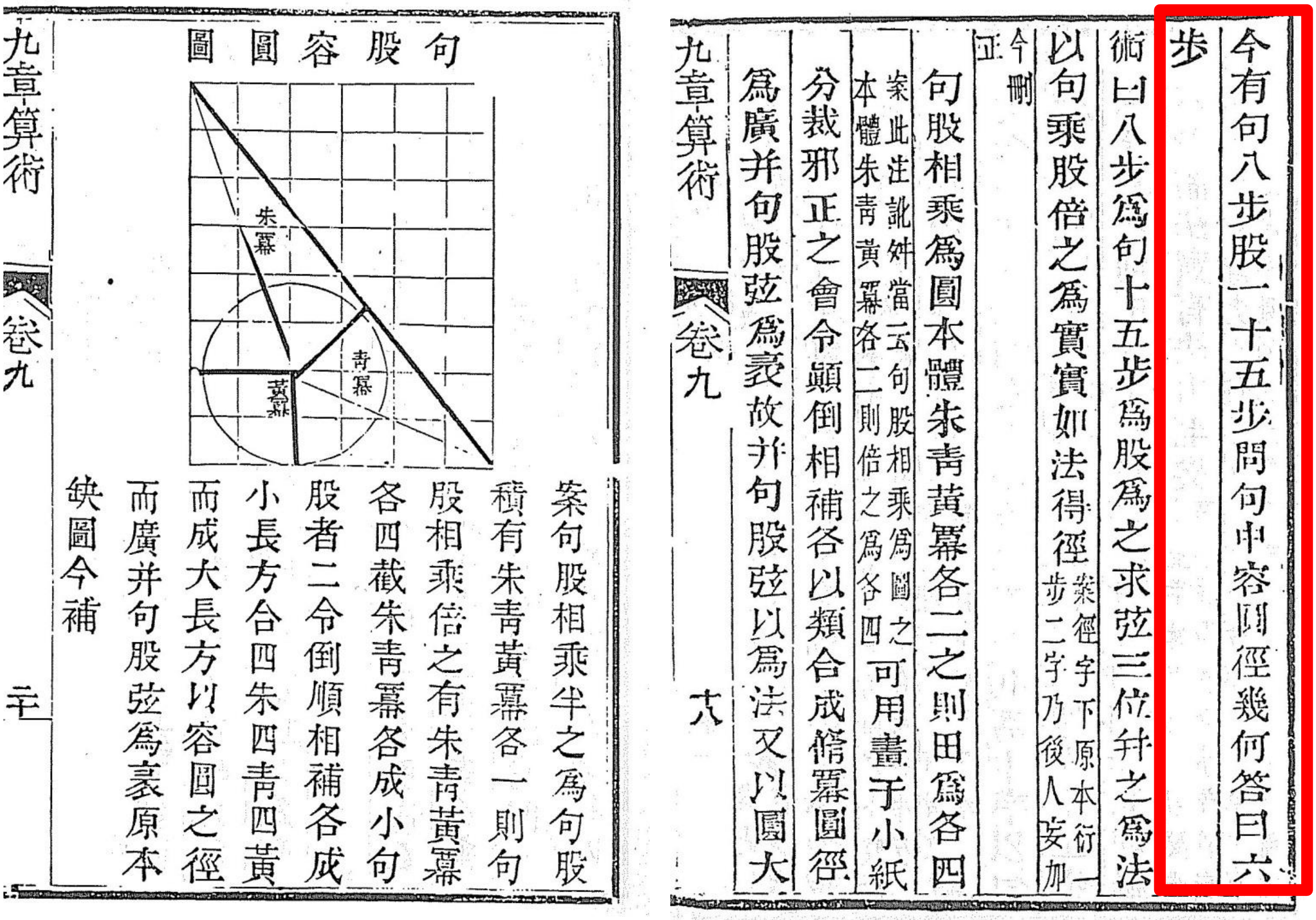
$$ab = (a + b)\,x$$

$$\boxed{x = \frac{ab}{a+b}}$$

http://ggbtu.be/m2812253

Method 2 (ratio and proportion)

— coming in a minute !



今有句八步股一十五步問句中容圓徑幾何答曰六步

術曰八步爲句十五步爲股爲之求弦三位并之爲法以句乘股倍之爲實實如法得徑案徑字下原本衍一步二字乃後人妄加今删正

句股相乘爲圓本體朱青黃冪各二之則田爲各四案此注訛舛當云句股相乘爲圖之本體朱青黃冪各二則倍之爲各四可用畫于小紙分裁邪正之會令顚倒相補各以類合成脩冪圓徑爲廣并句股弦爲袤故并句股弦以爲法又以圓大

九章算術 卷九 六

句股容圓圖

朱冪
青冪
黃冪

案句股相乘半之爲句股積有朱青黃冪各一則句股相乘倍之有朱青黃冪各四截朱青冪各成小句股者二令倒順相補各成小長方合四朱四青四黃而成大長方以容圓之徑而廣并句股弦爲袤原本缺圖今補

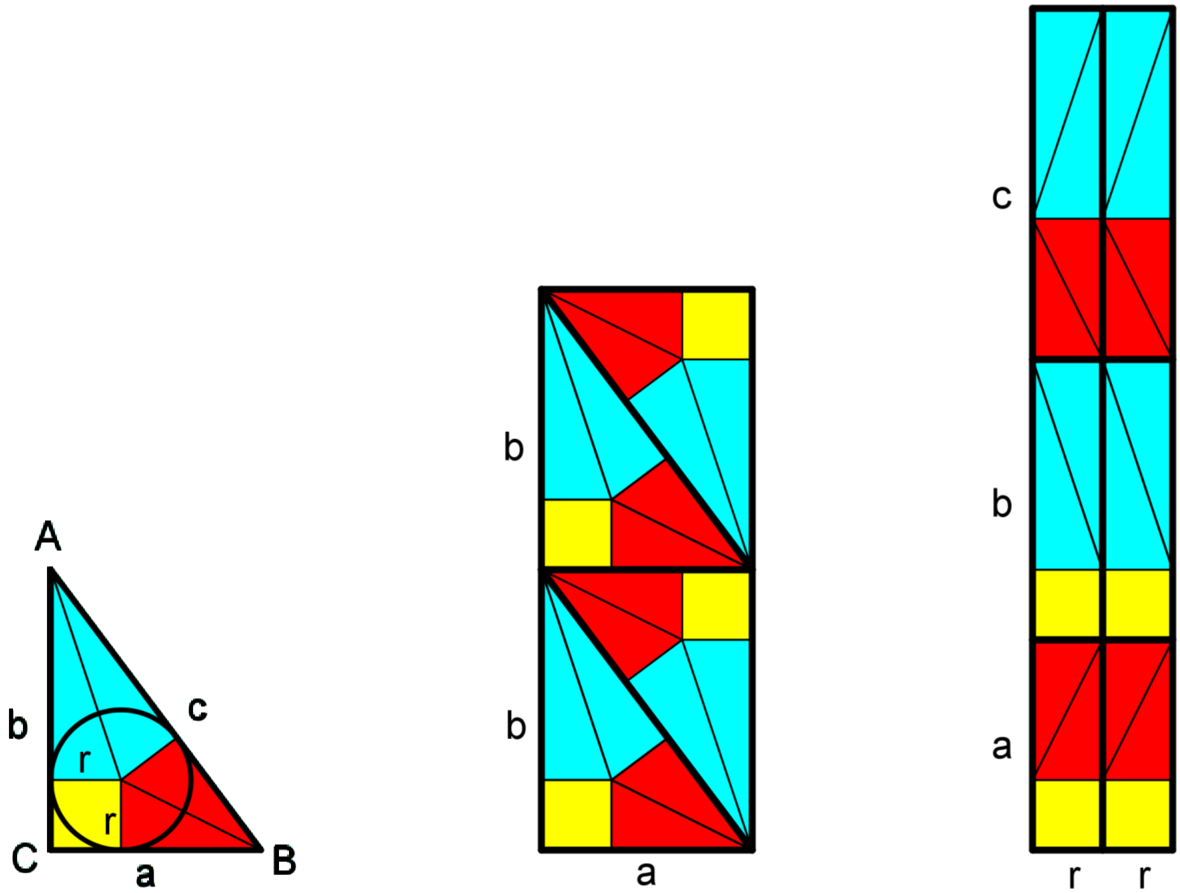
九章算術 卷九 七

Now given a right-angled triangle whose *gou* is 8 *bu* and whose *gu* is 15 *bu*. What is the diameter of its inscribed circle? The answer is 6 *bu*.

Jiuzhang Suanshu

[九章算術, compiled between 100 B.C.E. and 100 C.E.]

Chapter 9, Problem 16.

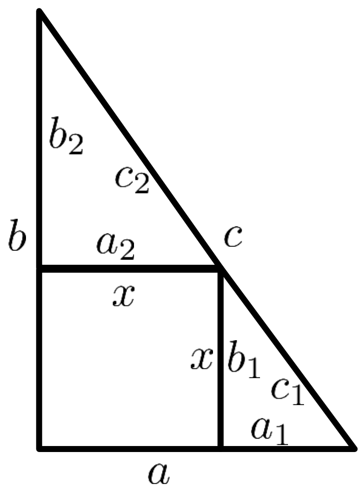


$$2r(a+b+c) = 2ab$$

$$\therefore \boxed{d = 2r = \frac{2ab}{a+b+c}}$$

http://ggbtu.be/m697695

Alternative proof of the formula in Problem 15 of Chapter 9 *of Jiuzhang Suanshu* (LIU Hui)



"To the top and to the right of the square there appear respective smaller right triangles. The relations between their sides retain the same rates as in the original triangle."

方在勾中，則方之兩廉各自成小勾股，而其相與之勢不失本率也。

$$a : b : c = a_1 : b_1 : c_1 = a_2 : b_2 : c_2.$$

$$\text{Hence, } \frac{a+b}{b} = \frac{a_1+b_1}{b_1} = \frac{a}{x},$$

$$\therefore \quad x = \frac{ab}{a+b}.$$

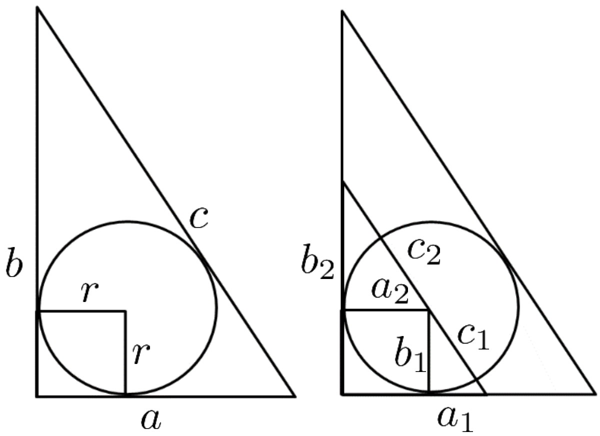
$$\left[\begin{array}{ll} \text{or} & \dfrac{a}{a+b} = \dfrac{a_2}{a_2+b_2} = \dfrac{x}{b}, \\ & \therefore \quad x = \dfrac{ab}{a+b}. \end{array} \right]$$

Exercise: Find a similar proof for the formula

$$d = \frac{2ab}{a+b+c}$$

in Problem 16 of Chapter 9 of *Jiuzhang Suanshu*.

Alternative proof of the formula in Problem 16 of Chapter 9 of *Jiuzhang Suanshu* (LIU Hui)



又畫中弦以相規會，則勾、股之面中央小勾股弦；勾之小股、股之小勾皆小方之面，皆圓徑之半。其數故可衰之。

"Draw a *zhong xian* [middle hypotenuse] through the centre to observe the availability of [useful] synthesis. There appear respective smaller right triangles. The smaller *gou* on the *gu* and the smaller *gu* on the *gou* are both sides of the small square, which is the radius of the inscribed circle. The method of *cui* [proportion] can be applied to the quantities."

$$a : b : c = a_1 : b_1 : c_1 = a_2 : b_2 : c_2.$$

$$\text{Hence, } \frac{a+b+c}{b} = \frac{a_1+b_1+c_1}{b_1} = \frac{a}{r}.$$

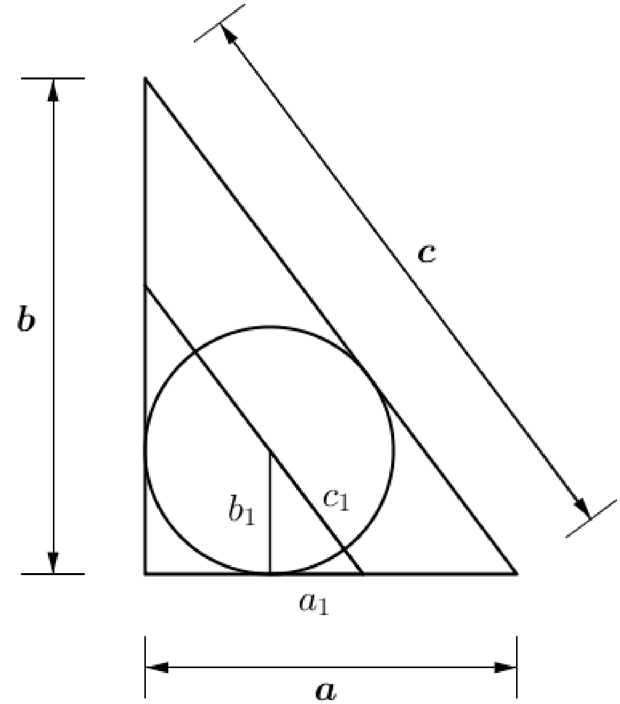
$$\therefore \quad r = \frac{ab}{a+b+c}, \quad d = \frac{2ab}{a+b+c}.$$

$$\left[\begin{array}{ll} \text{or} & \dfrac{a}{a+b+c} = \dfrac{a_2}{a_2+b_2+c_2} = \dfrac{r}{b}, \\ & \therefore \quad r = \dfrac{ab}{a+b+c}, \quad d = \dfrac{2ab}{a+b+c}. \end{array} \right]$$

Intriguing question: How did LIU Hui know that

$$a_1 + b_1 + c_1 = a, \;\; a_2 + b_2 + c_2 = b?$$

Did he understand the property of the sum of angles of a (right) triangle?

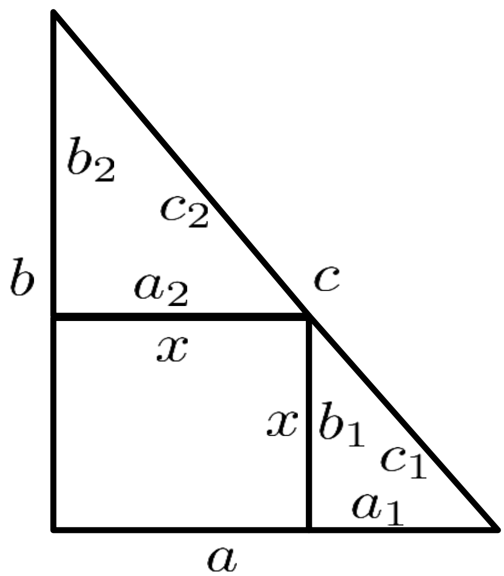


Why is

$a_1 + b_1 + c_1 = a$?

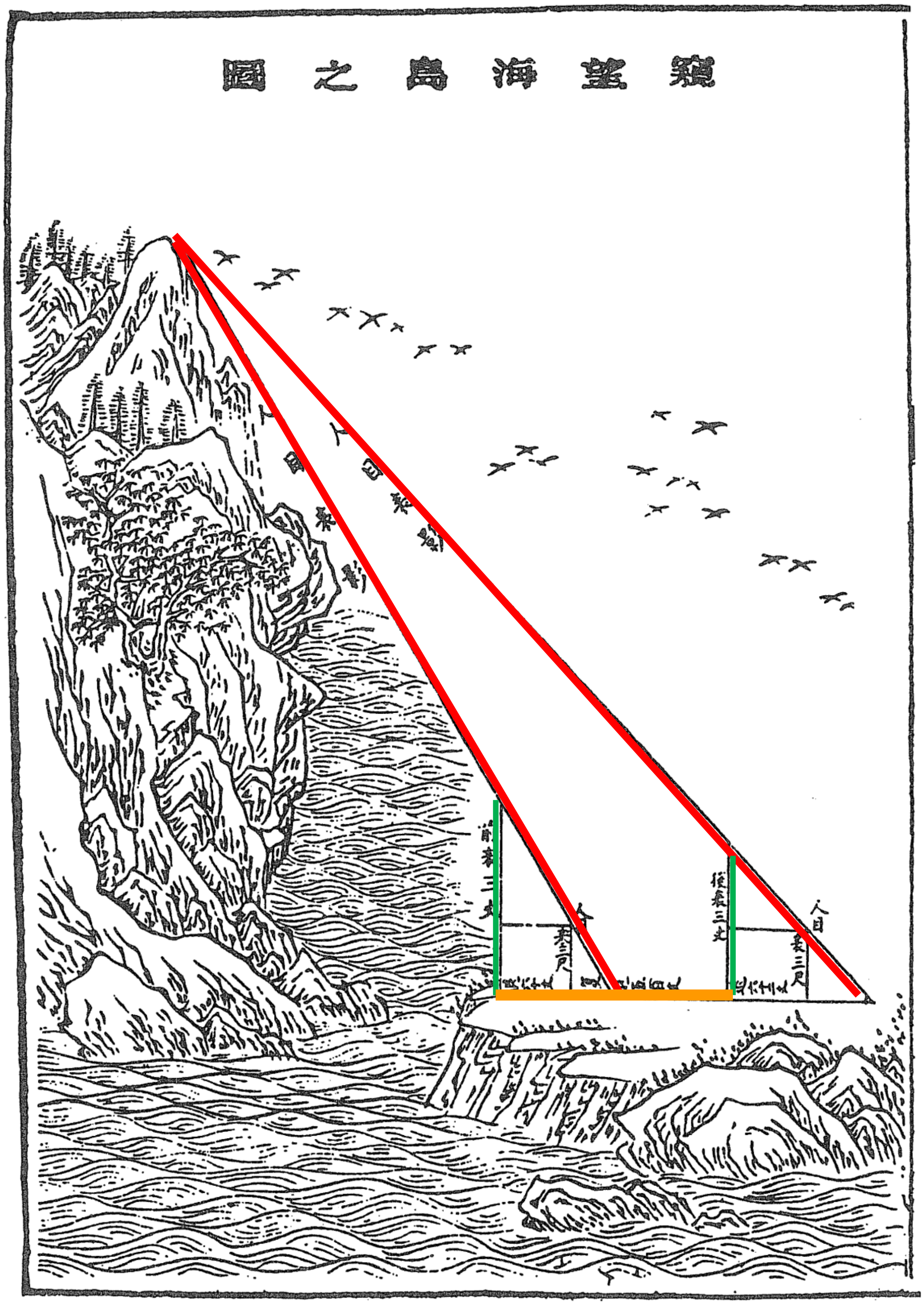
A plausible explanation?

http://ggbtu.be/m2811681



$$a : b : c$$
$$= a_1 : b_1 : c_1$$
$$= a_2 : b_2 : c_2$$

Today we see this relationship readily by the knowledge of similar triangles, but how would Chinese mathematicians in the past see it?



LIU Hui's Method of Double-Difference in *Haidao Suanjing* [海島算經 Sea Island Mathematical Manual] (3rd century) as illustrated in *Gujin Tushu Jicheng* [古今圖書集成 Complete Collection of Pictures and Writings of Ancient and Modern Times] (1726)

海島題解

魏劉徽注九章立重差著於勾股之下以闡世術夫度高測深勾股之法則無自而可知故重表累矩三望四望旁求審察是以松山高下方邑大小其重表也岸望谷深山望津廣其累矩也登望松高遙望波口非三望之術乎清淵白石登山臨邑非四望之術乎海島去表爲之篇首因以名之實九章勾股之遺法也迄今千餘載間唐李淳風而續算草未聞解白作法之旨者輝嘗置海島小圖於座右乃見先賢作法之萬一若欲盡傳豈不輕易祕旨或不傳流亦無以伸前賢之美本經題目廣遠難於引證學者非之今將孫子度影量竿題問引用詳解以驗小圖姑以一間其餘好學君子自能觸類而攷何必輕傳

凡股中容橫句中容直二積皆同古人以題易名若非釋名則無以知其源

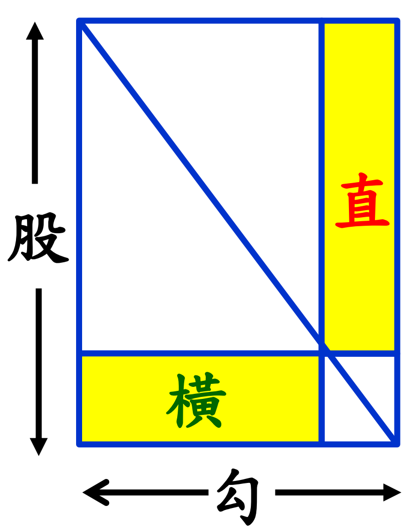
Explanation of the method of double difference (重差術) by LIU Hui (劉徽) by YANG Hui (楊輝) in 1275

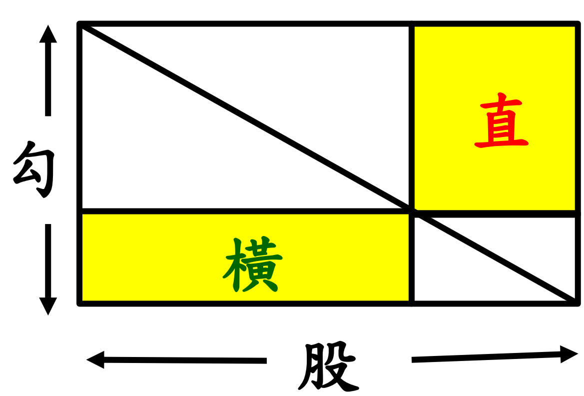
勾(股)中容横。股(勾)中容直。
二積皆同。古人以題易名。
若非釋名。則無以知其源。

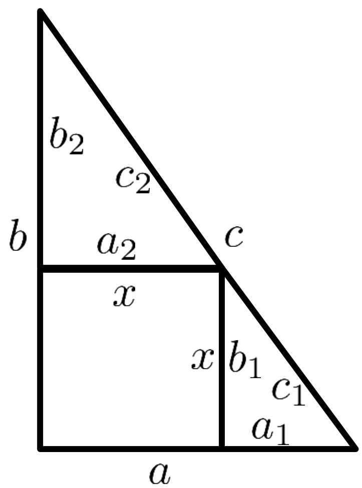
(The horizontal rectangle formed by part of the base and the vertical rectangle formed by part of the perpendicular are equal in area. Men of the past changed the names of their methods from problem to problem ...)

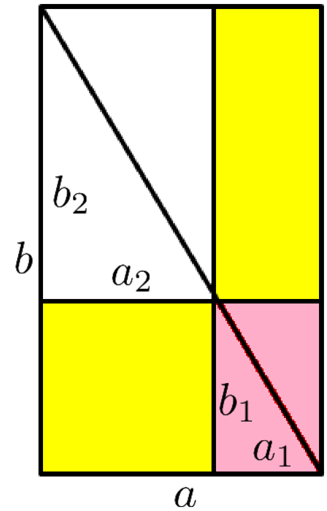
Compare with Proposition 43 Of Book I of Euclid's *Elements*.

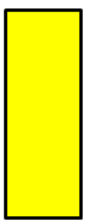
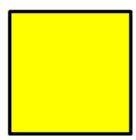
楊輝，《續古摘奇算法(卷下)》
YANG Hui, *Continuation of Ancient Mathematical Methods for Elucidating the Strange* [*Properties of Numbers*] (Chapter II) (1275)



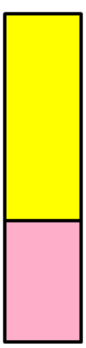
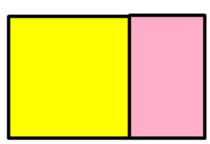






 $= a_1 b_2,$  $= a_2 b_1.$

Hence, $a_1 b_2 = a_2 b_1$, or $a_1 : a_2 = b_1 : b_2$.

 $= a_1 b,$  $= a b_1.$

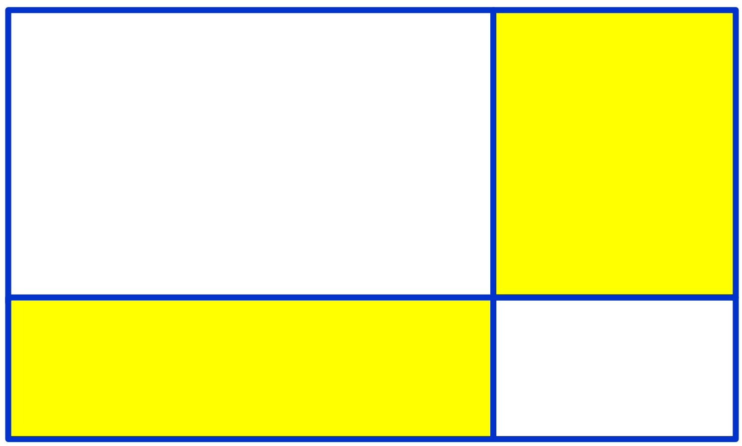
Hence, $a_1 b = a b_1$, or $a : a_1 = b : b_1$.

Since $c^2 = a^2 + b^2, c_1^2 = a_1^2 + b_1^2, c_2^2 = a_2^2 + b_2^2,$

we have $a : a_1 : a_2 = b : b_1 : b_2 = c : c_1 : c_2,$

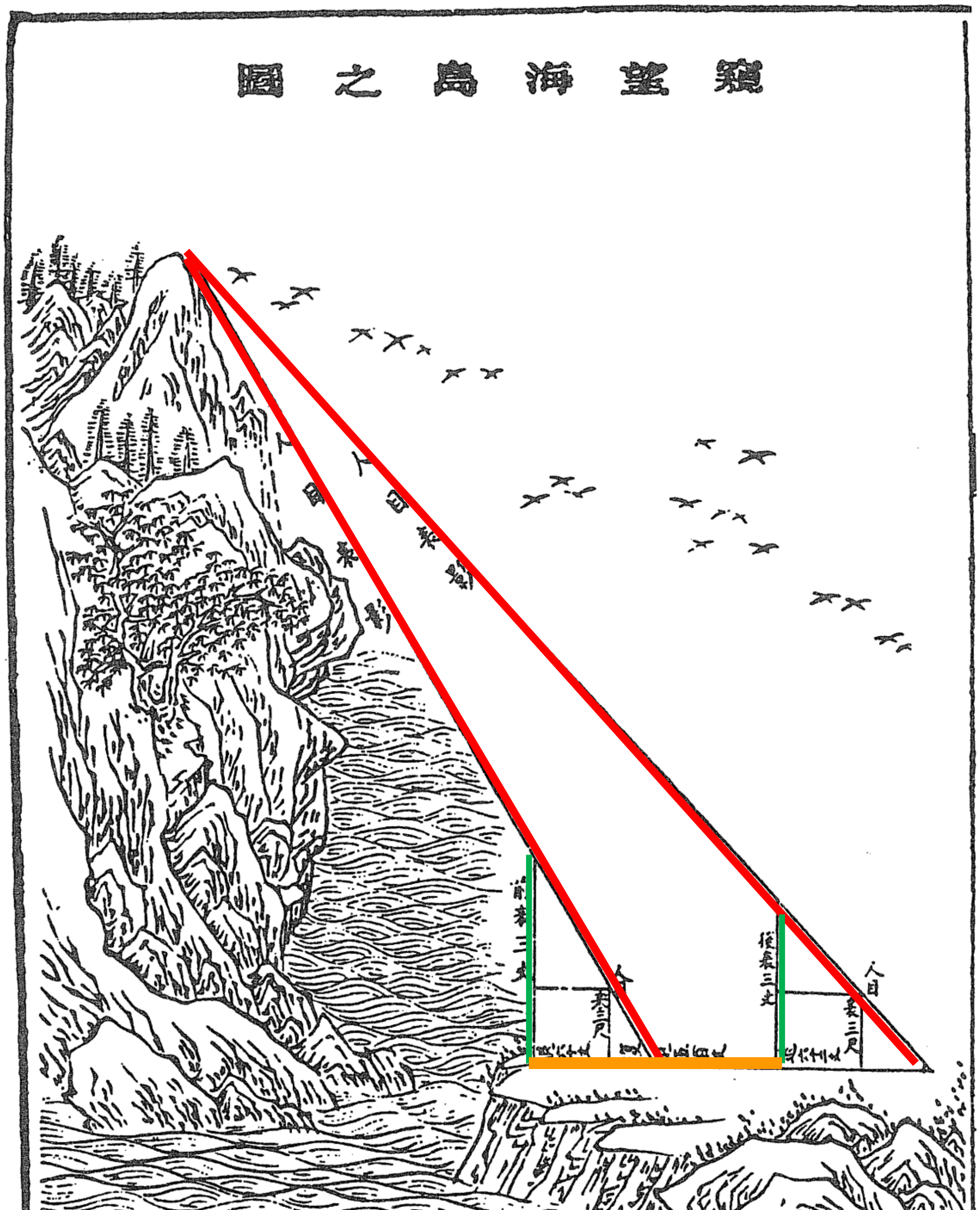
or $\quad a : b : c = a_1 : b_1 : c_1 = a_2 : b_2 : c_2.$

A pedagogical extension to a locus problem (but with no historical context)



Question: When (and only when) will the two regions have equal area?

http://ggbtu.be/m2467811



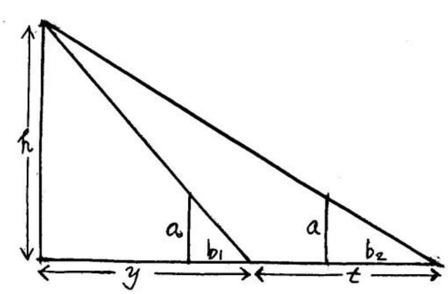
LIU Hui's Method of Double-Difference in *Haidao Suanjing* [海島算經 Sea Island Mathematical Manual] (3rd century) as illustrated in *Gujin Tushu Jicheng* [古今圖書集成 Complete Collection of Pictures and Writings of Ancient and Modern Times] (1726)

Aryabhata I
阿耶波多
(c.476-550)

Aryabhatiya
《阿耶波多曆數表》

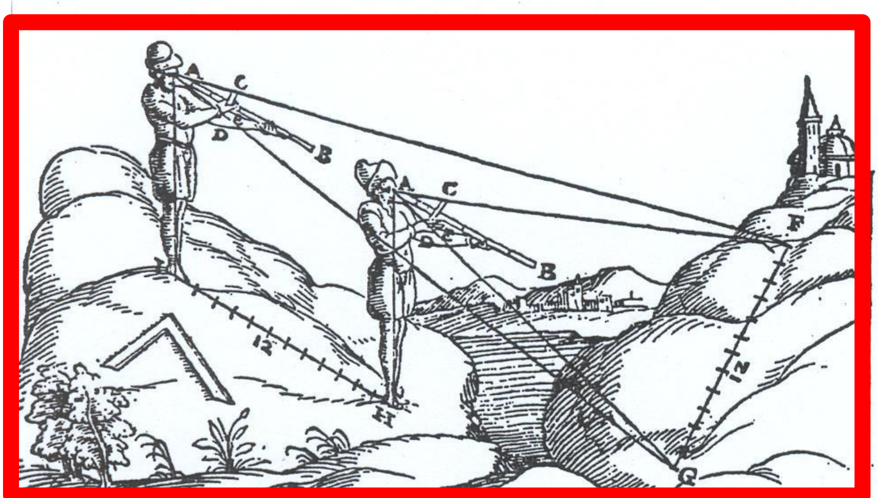
Book II, Stanza 16

The distance between the ends of the two shadows multiplied by the length of the first shadow and divided by the difference in length of the two shadows gives the *kotī* . The *kotī* multiplied by the length of the gnomon and divided by the length of the (first) shadow gives the length of the *bhujā* .



$$y = \frac{tb_1}{b_2 - b_1}$$

$$h = \frac{ya}{b_1} \quad \left(= \frac{ta}{b_2 - b_1}\right)$$

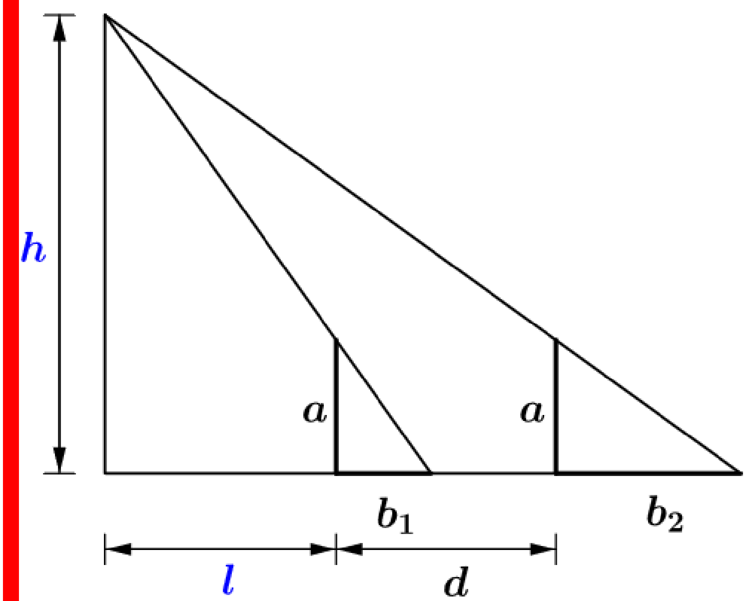


Orence Fine, De re & praxi geometrica (1556)

John Sellers, *Practical Navigation* (1672)

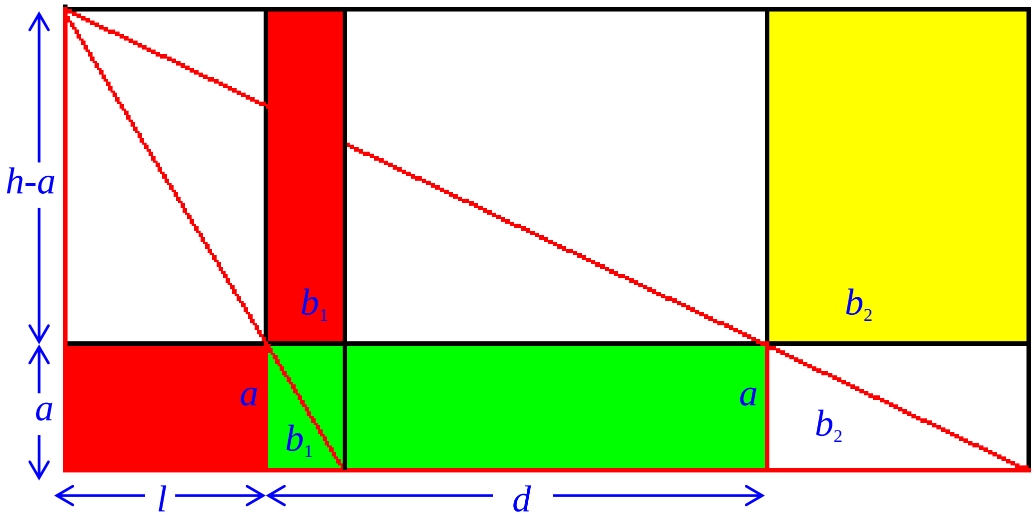
The invention of the cross-staff (or Jacob's staff) has been credited to Levi ben Gerson (1288-1344).

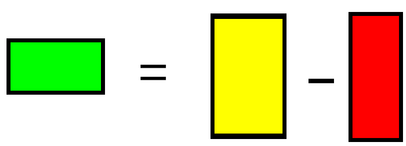
Given a, d, b_1 and b_2, how can we express h and l in terms of a, d, b_1 and b_2 ?



http://ggbtu.be/m2812113

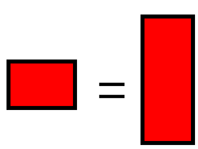
Explanation of the formulae by YANG Hui (1275)





$$ad = b_2(h-a) - b_1(h-a)$$
$$= (b_2 - b_1)(h-a)$$

$$h = \frac{ad}{b_2 - b_1} + a$$



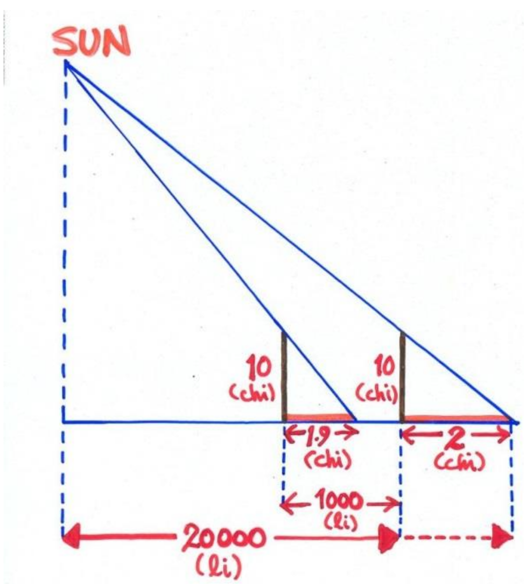
$$la = b_1(h-a) = \frac{b_1 ad}{b_2 - b_1}$$

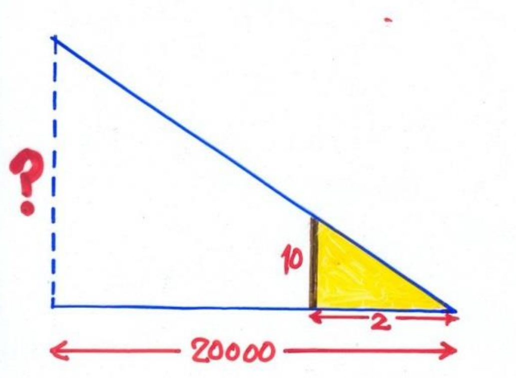
$$l = \frac{b_1 d}{b_2 - b_1}$$

Huainanzi (淮南子), *Tianwenxun* (天文訓)

[The Book of the Prince of Huai Nan, Chapter 3: Treatise On the Pattern of Heaven], 2nd century B.C.

"To find the height of heaven, set up [two] gnomons ten *chi* high and 1000 *li* apart due north-south. Measure their shadows (at noon) on the same day…."





Go to the south for 1000 *li*, the shadow diminishes by 0.1 *chi*.

Go to the south for 1 *li*, the shadow diminishes by 0.1/1000 *chi*.

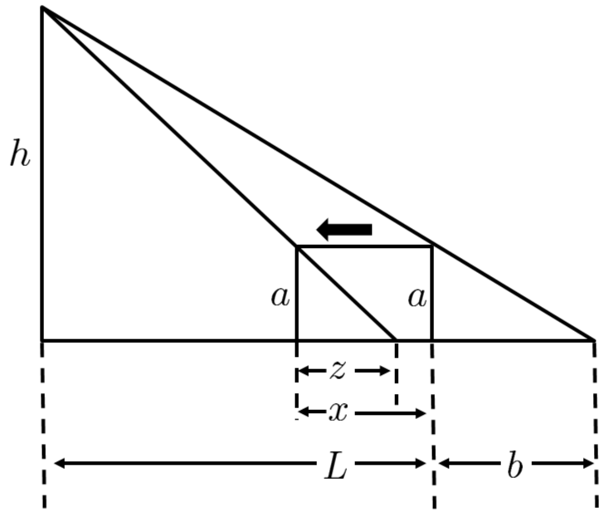
Go to the south for 20,000 *li*, the shadow diminishes to zero *chi*.

2 : 10 = 1 : 5

= 20,000 : ?

? = 100,000

[should be
2 : 10 = 20000+2/1800 : ?]



x	z	$y = b - z$
0	b	0
x_1	z_1	$b - z_1$
x_2	z_2	$b - z_2$
$\vdots$	$\vdots$	$\vdots$
L	0	b

Does *y* (shortening of the shadow) vary linearly with *x* (distance moved by the gnomon) ?

$$\frac{b}{a} = \frac{L+b}{a} \text{ and } \frac{b-y}{a} = \frac{(L-x)+(b-y)}{h}.$$

$$\text{Hence, } \frac{b}{a} - \frac{y}{a} = \frac{L+b}{h} - \frac{x}{h} - \frac{y}{h},$$

$$\frac{y}{a} = \frac{x}{h} + \frac{y}{h},$$

$$\text{therefore, } y = \frac{ax}{h-a}.$$

YES , *y* varies linearly with *x* .

How does the shortening of the shadow vary with the distance moved by the gnomon?

http://ggbtu.be/m2860149

ZHAO Youqin (趙友欽), *Gexiang Xinshu* [革象新書 New Writing on the Images of Alteration], Chapter 5, Section on Measurement of Heaven with *Gougu* (*ca*. 1280)

- **Alexi Volkov, The mathematical work of Zhao Youqin: Remote surveying and the computation of *π*, *Taiwanese Journal for Philosophy and History of Science*, 5 (1) (1996).**

Method of extracting square root

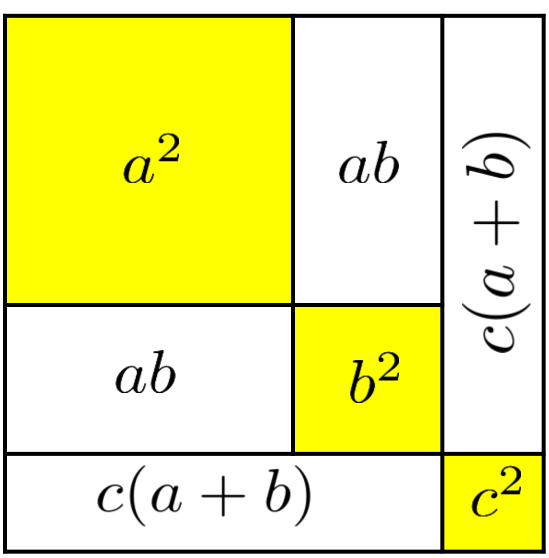
http://ggbtu.be/m2744339

《九章算術》*Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art), compiled between 1st century B.C.E. and 1st century.

Chapter 4 (Short Width) 少廣
Problem 12

Now given an area 55225 [square] *bu*. Tell: what is the side of the square?

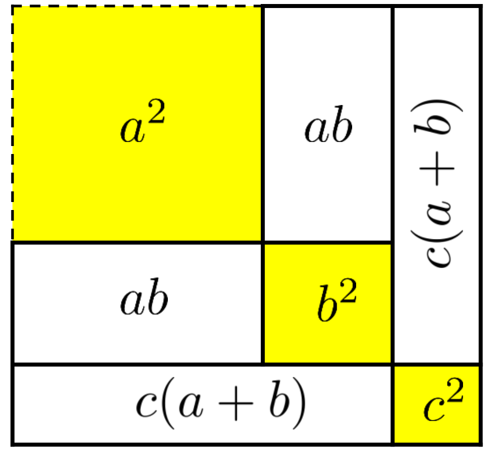
…… The Rule for Extracting the Square Root: Lay down the given area as *shi*. Borrow a counting rod to determine the digit place. Set it under the unit place of the *shi*. Advance [to the left] every two digit places as one step. Estimate the first digit of the root. ……



$$(a+b+c)^2 = 55225$$
$$a \in \{0, 100, 200, \ldots, 900\}$$
$$b \in \{0, 10, 20, \ldots, 90\}$$
$$c \in \{0, 1, 2, \ldots, 9\}$$

$a = 200 \qquad a^2 = 40000$

$55225 - 40000 = 15225$

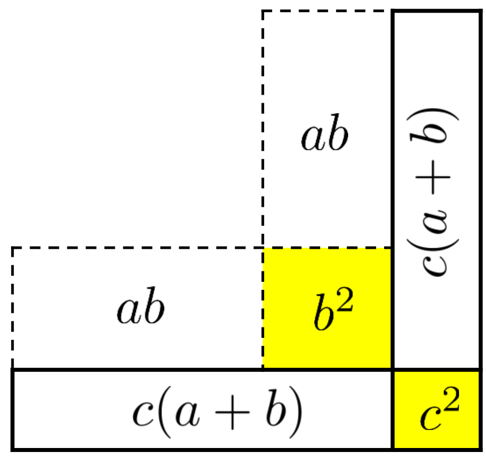


$b = 30 \qquad b^2 + 2ab = 12900$

$15225 - 12900 = 2325$

$c = 5 \qquad c^2 + 2c(a+b) = 2325$

$2325 - 2325 = 0$



$$x = 200 + 30 + 5 = 235$$

	2	3	5
	5	52	25
2	4		
43	1	52	
	1	29	
465		23	25
		23	25

	2	3	5	•	0	3	
	5	52	40	•	00	00	…
2	4						
	1	52					
4 3	1	29					
		23	40				
4 6 5		23	25				
			15		00		
4 7 0 0			00		00		
			15		00	00	
4 7 0 0 3			14		10	09	
					89	91	…
…							

Chapter 4 (Short Width) 少廣
Problem 12

Now given an area 55225 [square] *bu*. Tell: what is the side of the square?

…… The Rule for Extracting the Square Root: Lay down the given area as *shi*. Borrow a counting rod to determine the digit place. Set it under the unit place of the *shi*. Advance [to the left] every two digit places as one step. Estimate the first digit of the root. ……

If there is a remainder, [the number] is called unextractable, it should be defined as the side on which the square has the area of the *shi* [若開之不盡者，為不可開，當以面命之。]. ……

$\sqrt{A} = a + \cdots$, a is the nearest integer to $\sqrt{A}$.

$$a + \frac{A - a^2}{2a+1} < \sqrt{A} < a + \frac{A - a^2}{2a}$$

開之不盡者爲不可開當以面命之

術或有以借筭加定法而命分者雖粗相近不可用也凡開積爲方方之自乘當還復其積分令不加借筭而命分則常微少其加借筭而命分則又微多其數不可得而定故惟以面命之爲不失耳譬猶以三除十以其餘爲三分之一而復其數可以舉不以面命之加定法如前求其微數微數無名者以爲分子其一退以十爲母其再退以百爲母退之彌下其分彌細則朱冪雖有所棄之數不足言之也

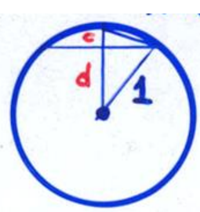
若實有分

弦半面五寸爲句爲之求股以句冪二十五寸減弦冪餘七十五寸開方除之下至秒忽又一退法求其微數微數無名知以爲分子以~~下~~十爲分母約作五分忽之二故得股八寸六分六釐二~~絲~~秒五忽五分忽之二以減半徑餘一寸三分三釐九毫七秒四忽五分忽之三謂之小句小句知半面五寸之句弧之半面而又謂之小股爲之求弦其冪二千六百七十九億四千九百一十九萬三千四百四十五忽全分并之

《九章算術》
第四章(少廣)
第十六題

《九章算術》
第一章(方田)
第三十二題

[the number] is called unextractable, it should be defined as the side on...



$d=0.866025\ldots$

$c=1-d=0.133974\ldots$

十	百	千	萬	
10	10^{2}	10^{3}	10^{4}	
億	兆	京		
10^{8}	10^{12}	10^{16}		
垓	秭	穰	溝	澗
10^{20}	10^{24}	10^{28}	10^{32}	10^{36}
正	載	極	恒河沙	
10^{40}	10^{44}	10^{48}	10^{52}	
阿僧祇	那由他	不可思議		
10^{56}	10^{60}	10^{64}		
無量	大數			
10^{68}	10^{72}			

分	釐	毫	絲	忽
10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
微	纖	沙	塵	埃
10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}

渺	漠	模糊	逡巡
10^{-11}	10^{-12}	10^{-13}	10^{-14}
須臾	瞬息	彈指	剎那
10^{-15}	10^{-16}	10^{-17}	10^{-18}

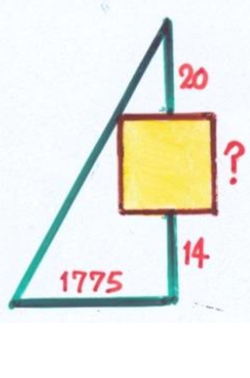
《九章算術》卷九題二十

今有邑方不知大小，各中開門。出北門二十步有木。出南門十四步，折而西行一千七百七十五步見木。問邑方幾何。

荅曰：二百五十步。

術曰：以出北門步數乘西行步數，倍之，為實。此以折而西行為股，自木至邑南十四步為句，以出北門二十步為句率，北門至西隅為股率，即半廣數。故以出北門句率乘西行股，得半廣股率乘句之冪。然此冪居半以西，故又倍之，合半以東也。并出南門步數為從法，開方除之，即邑方。此術之冪，東西如邑方，南北自木盡邑南十四步。故連四步為袤，合南北步數為廣袤差，故并兩步數為從法，以為隅外之冪也。

帶從開方法



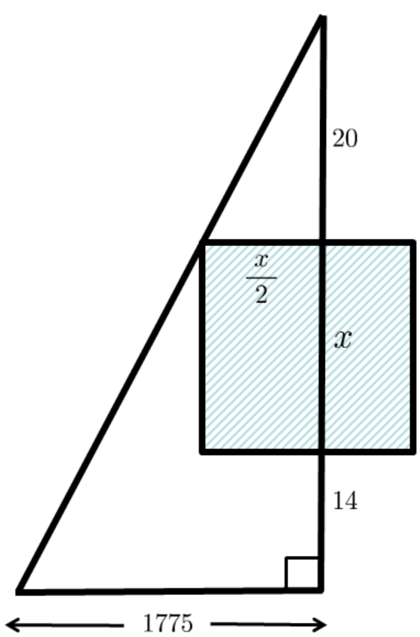
$$(?)(?+34)=2\times 20\times 1775$$

《九章算術》*Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art), compiled between 1st century B.C.E. and 1st century C.E.

Chapter 9 (Right-angled triangles) 勾股

Problem 20

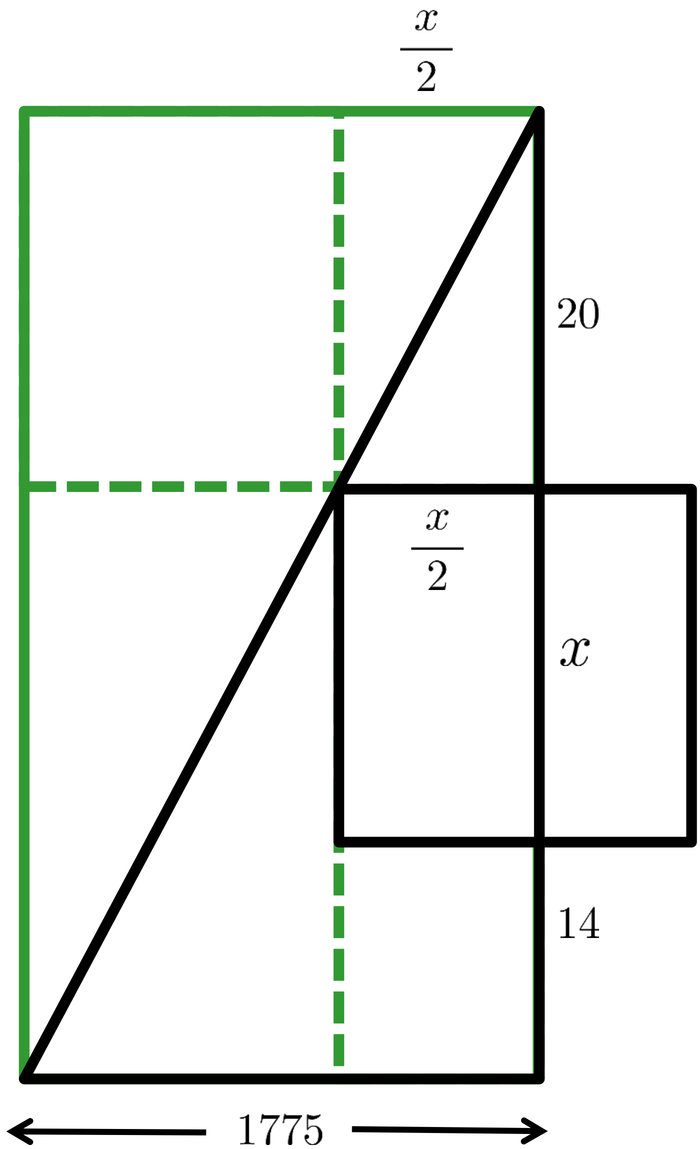
Now given a square city of unknown side, with gates opening in the middle. 20 *bu* from the north gate there is a tree, which is visible when one goes 14 *bu* from the south gate and then 1775 *bu* westward. Tell: what is the length of each side?



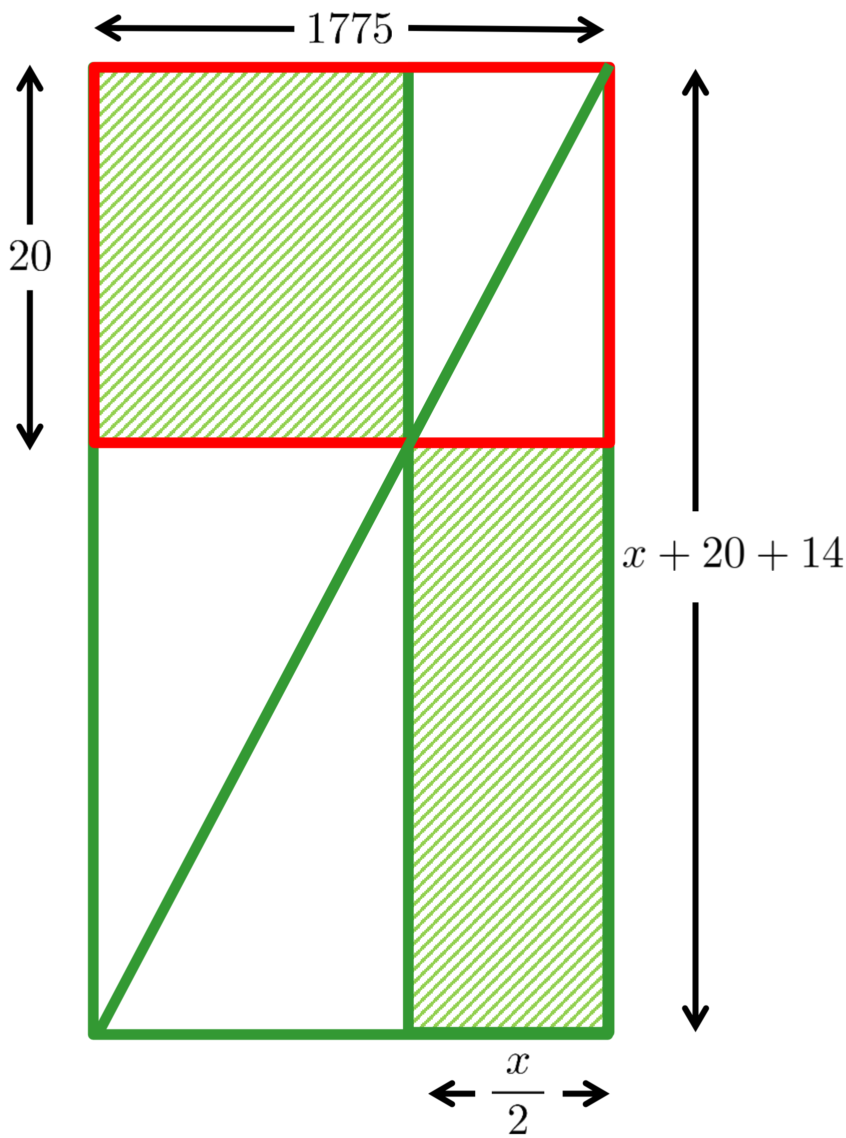
$$\boxed{x^2 + 34x = 71000}$$

$$\boxed{\begin{aligned} x(x+34) &= 2 \times 20 \times 1775 \\ &= 71000 \end{aligned}}$$

Extraction of square root with accompany number (帶從開方法)



$\frac{x}{2}$
20
$\frac{x}{2}$
x
14
1775

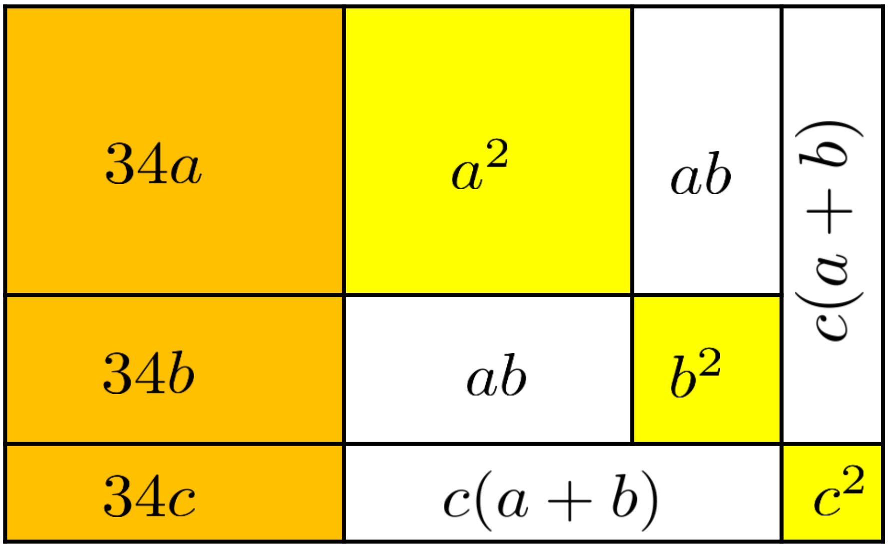


$$\frac{x}{2}(x+20+14) = 20 \times 1775$$

$$x(x+34) = 71000$$

Expressed as a quadratic equation in today's textbook, it reads

$$\boxed{x^2 + 34x - 71000 = 0.}$$



$$(a+b+c)^2 + 34(a+b+c) = 71000$$
$$a \in \{0, 100, 200, \ldots, 900\}$$
$$b \in \{0, 10, 20, \ldots, 90\}$$
$$c \in \{0, 1, 2, \ldots, 9\}$$

$$a = 200 \qquad a^2 + 34a = 46800$$
$$71000 - 46800 = 24200$$

$$b = 50 \qquad b^2 + 2ab + 34b = 24200$$
$$24200 - 24200 = 0$$

$$c = 0$$

$$x = 200 + 50 + 0 = 250$$

Problem 1 in Chapter 7, *Excess and Deficit* [盈不足] of *Jiuzhang Suanshu* [九章算術, Nine Chapters on the Mathematical Art], compiled between 1st century B.C.E. and 1st century.

九章算術卷第七　算經十書
魏　劉　徽　注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉　勅注釋
盈不足以御隱雜互見
今有共買物人出八盈三人出七不足四問人
數物價各幾何
荅曰七人
物價五十三

盈不足按盈者謂之朓不足者謂之朒所出率謂之假令盈朒維乘兩設者欲爲齊同之意
術曰盈不足相與同共買物者置所出率
盈不足各居其下令維乘所出率并以爲
實并盈不足爲法據共買物人出八盈三人出七不足四齊其假令同其盈朒盈朒俱十二通計齊則不盈不朒之正數故可并以爲實并盈不足爲法齊之三十二者是四假令有盈十二齊之二十一者是三假令亦朒十二并七假令合爲一實故并三四爲法有分者通之若兩設有分者齊其子同其母此問兩設俱見零分故齊其子同其母副置所出率以少
減多餘以約法實實爲物價法爲人數令下維乘上訖以同約之不可約故以乘同之所出率以少減多者餘謂之設差以爲少設則并盈朒是爲定實故以少設約法則爲人數約實則爲物價盈朒當與少設相通不可偏約亦當分母乘設差爲約法實
其一術曰并盈不足爲實以所出率以少
減多餘爲法實如法得一人以所出率乘
之減盈增不足卽物價此術意謂盈不足爲衆人之差以所出率以少減多餘爲一人之差以一人之差約衆人之差故得人數也

This is known in the Western world as "rule of *khitai* (契丹算法? or is it a linguistic misunderstanding?)".

今有共買物，人出八，盈三;人出七，不足四。問:人數、物價各幾何？

《九章算術》第七章(盈不足)第一題

N = 人數， S = 物價．

a = 所出，得 e = 盈．

b = 所出，得 d = 不足．

$$aN = S + e,$$
$$bN = S - d.$$

故有

$$(a - b)N = e + d,$$
$$N = (e + d)/(a - b),$$

而 $S = aN - e$

$$= (ad + be)/(a - b).$$

今有共買物，人出八，盈三;人出七，不足四。問:人數、物價各幾何？

《九章算術》第七章(盈不足)第一題

N = 人數， S = 物價.

a = 所出，得 e = 盈.

b = 所出，得 d = 不足.

$$aN = S + e,$$

$$bN = S - d.$$

此乃今天初中學生熟悉的方法，在二千多年前，符號運算猶未發展起來，古人循何思路解決問題呢?

《九章算術》*Jiuzhang Suanshu*

(Nine Chapters on the Mathematical Art), compiled between 1st century B.C.E. and 1st century C.E.

Problem 12, Chapter 7

(Excess and Deficit 盈不足)

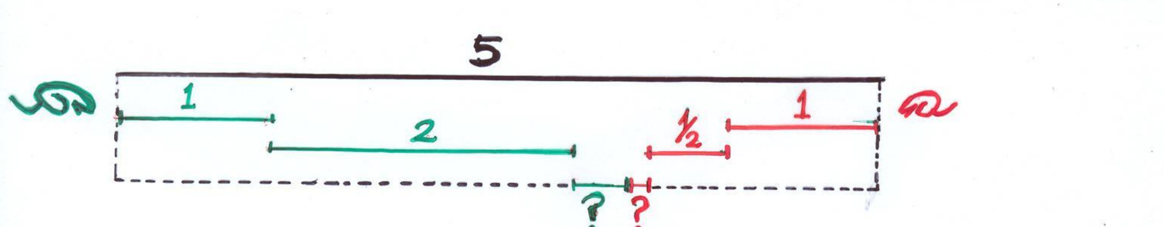
今有垣厚五尺兩鼠對穿大鼠日一尺小鼠亦日一尺大鼠日自倍小鼠日自半問幾何日相逢各穿幾何

荅曰二日十七分日之二

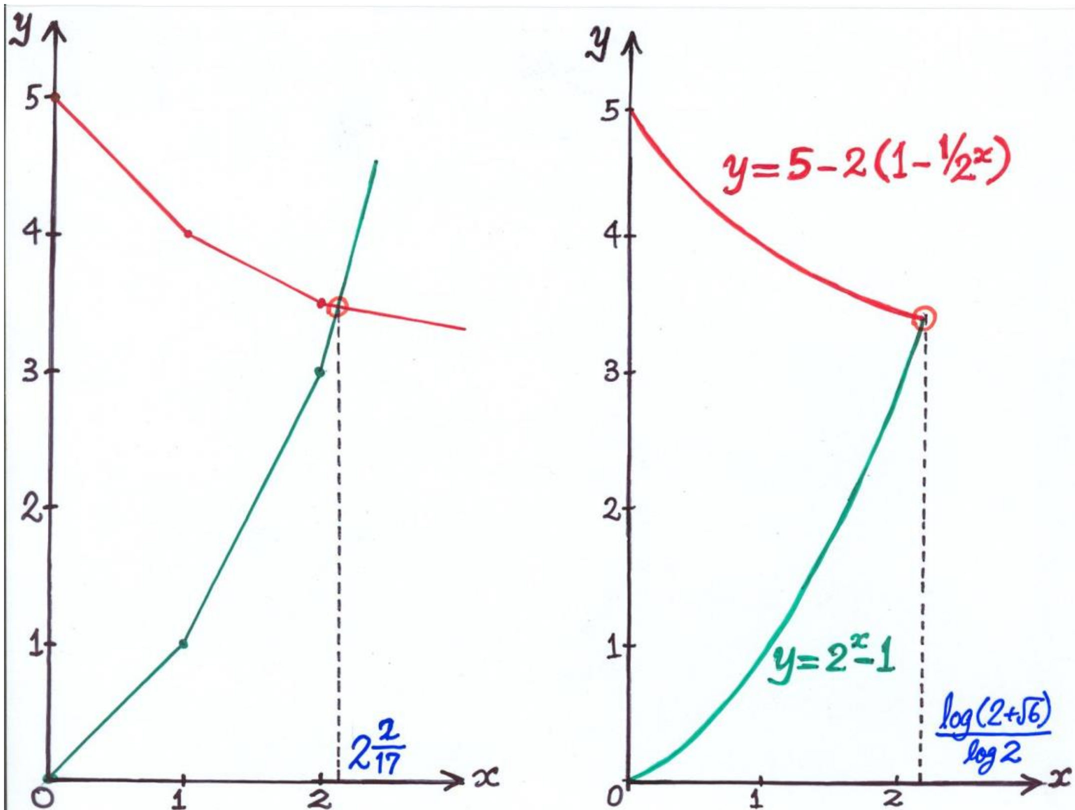
大鼠穿三尺四寸十七分寸之十二

小鼠穿一尺五寸十七分寸之五

術曰假令二日不足五寸令之三日有餘三尺七寸半大鼠日倍二日合穿三尺小鼠日自半合穿一尺五寸并大鼠所穿合尺五寸課於垣厚五尺是爲不足五寸令之三日大鼠穿得七尺小鼠穿得一尺七寸半并之以減垣厚五尺有餘三尺七寸半以盈不足術求之即得以後一日所穿乘日分子如日分母而一即各得日分子之中所穿故各增二日定穿即合所問也



http://ggbtu.be/3507645



$2^x - 1 = 5 - 2\,(1 - 1/2^x)$.

$2^{2x} - 4 \times 2^x = 2$.

Let $z = 2^x$, then $z^2 - 4z - 2 = 0$.

Take the positive root $z = 2 + \sqrt{6}$.

Hence $x = \log\,(2 + \sqrt{6})\,/\,\log 2$

$= 2.15363986 \ldots$.

[Compare with $2\frac{2}{17} = 2.11764705\ldots.$]

An unfair assessment?

It is an assessment made out of historical context!

今有共買物，人出八，盈三;人出七，不足四。問:人數、物價各幾何？

《九章算術》第七章(盈不足)第一題

N	物價(人出8)		物價(人出7)		物價相差
1	5	8	11	7	- 6
2	13	16	18	14	- 5
3	21	24	25	21	- 4
•	•	•	•	•	•
•	•	•	•	•	•
7	53	56	53	49	0
•	•	•	•	•	•
10	77	80	74	70	+ 3
11	85	88	81	77	+ 4
12	93	96	88	84	+ 5

「此術意謂盈不足為眾人之差，以所出率以少減多，餘為一人之差。以一人之差約眾人之差，故得人數也。」

今有共買物，人出八，盈三;人出七，不足四。問:人數、物價各幾何？

《九章算術》第七章(盈不足)第一題

「此術意謂盈不足為眾人之差，以所出率以少減多，餘為一人之差。以一人之差約眾人之差，故得人數也。」

$$aN = S + e,$$

$$bN = S - d.$$

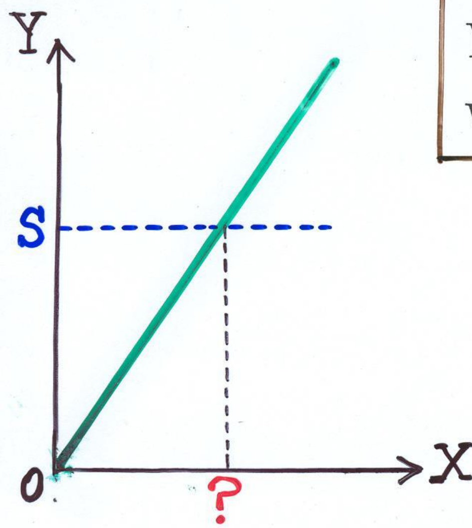
故有 $(a - b)N = e + d$,

即得

$$N = (e + d)/(a - b),$$

而 $S = aN - e$

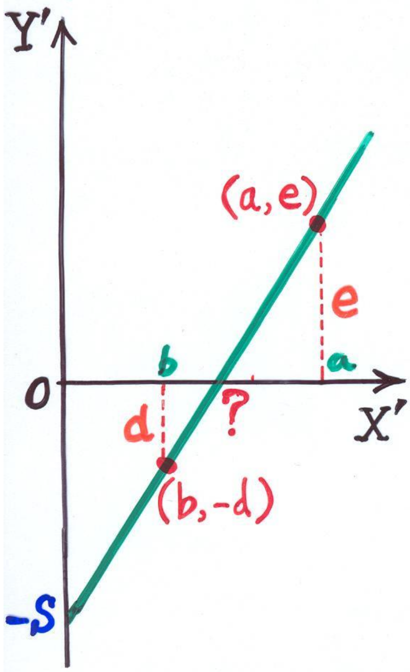
$$= (ad + be)/(a - b).$$



$Y = AX$

What X makes $Y = S$?

$Y' = Y - S$
$X' = X$



$Y' = AX' - S$

What X' makes $Y' = 0$?

Known: (a, e) and $(b, -d)$ are on the line, $a \neq b$.

$e = Aa - S,$

$-d = Ab - S.$

Hence $A = \dfrac{e+d}{a-b}$, $S = \dfrac{ad+be}{a-b}$

$$? = \frac{S}{A} = \frac{ad+be}{e+d}.$$

Method of double false position

$\longrightarrow$ **Method of interpolation**

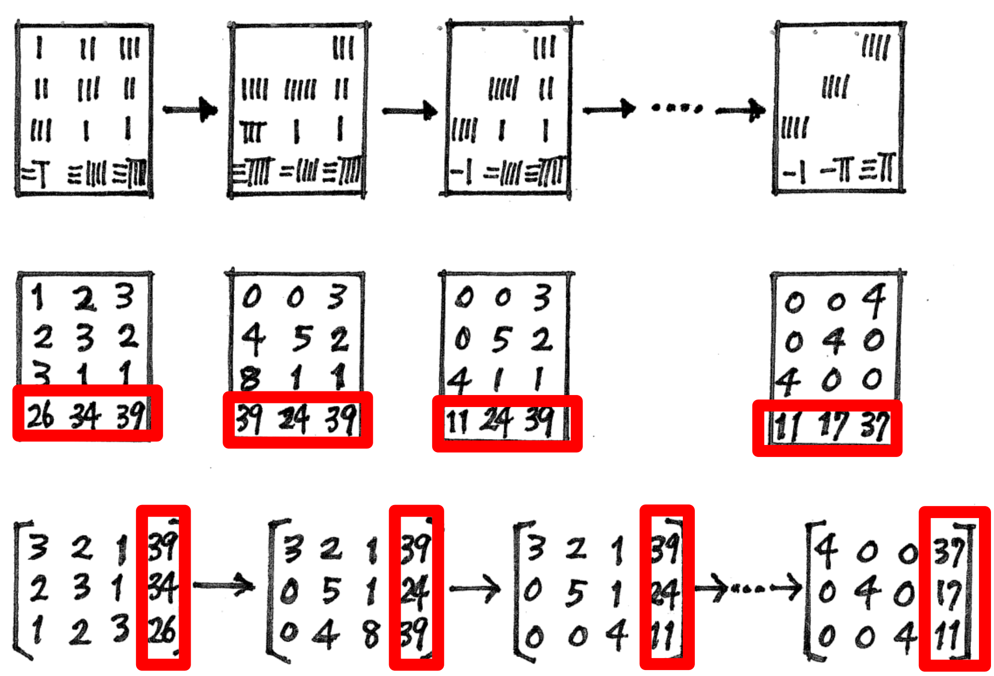
《九章算術》*Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art), compiled between 1st century B.C.E. and 1st century.

Problem 1, Chapter 8 (Rectangular Arrays 方程)

Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grade yield?

modern version

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned}$$



Gaussian elimination method

System of Linear Equations
Chapter 8 of *Jiuzhang Suanshu* 九章算術, *ca.* 1st century B.C.E. to 1st century C.E.

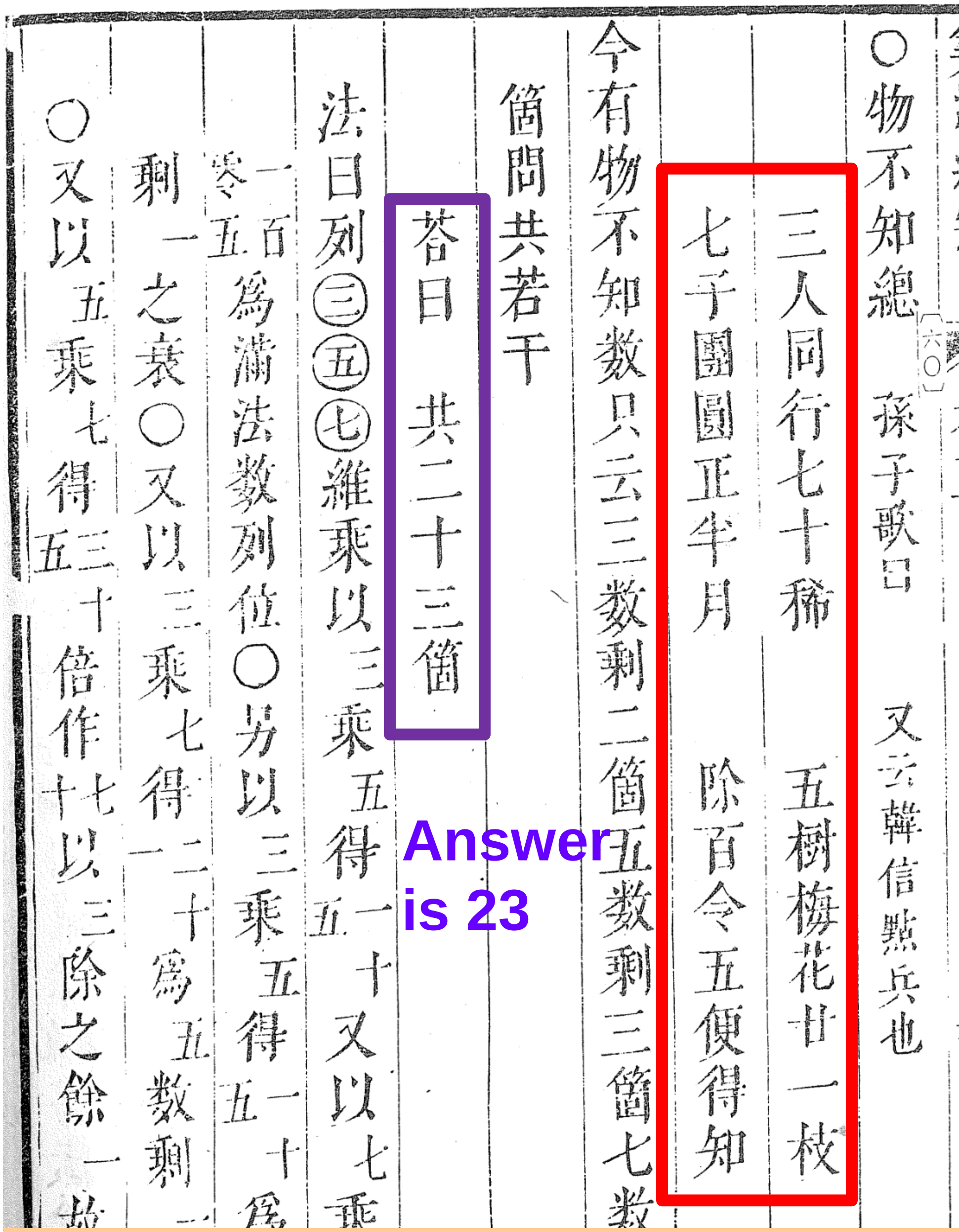
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= C_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= C_2 \\
&\cdots\cdots \\
a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n &= C_k
\end{aligned}
$$

System of Linear Congruence Equations
***Sunzi Suanjing* 孫子算經, 4th/5th century and *Shushu Jiuzhang* 數書九章, 1247.**

$$
\begin{aligned}
x &\equiv A_1 \pmod{m_1} \\
x &\equiv A_2 \pmod{m_2} \\
&\cdots\cdots \\
x &\equiv A_k \pmod{m_k}
\end{aligned}
$$

今有物不知其數三三數之賸二五五數之賸
三七七數之賸二問物幾何
荅曰二十三
術曰三三數之賸二置一百四十五五數
之賸三置六十三七七數之賸二置三十
并之得二百三十三以二百一十減之即
得凡三三數之賸一則置七十五五數之
賸一則置二十一七七數之賸一則置十
五一百六以上以一百五減之即得

Sunzi Suanjing **《孫子算經》**
[Master Sun’s Mathematical Manual]
4th/5th century.


○物不知總 孫子歌曰 又云韓信點兵也
三人同行七十稀 五樹梅花廿一枝
七子團圓正半月 除百令五便得知
今有物不知数只云三三数剩二箇五五数剩三箇七七数
箇問共若干
荅曰 共二十三箇
法曰列㊂㊄㊆維乘以三乘五得一十五又以七乘
一百零五為滿法数列位○另以三乘五得一十五一十為
剩一之衰○又以三乘七得二十一為五数剩一
○又以五乘七得三十五以三除之餘一倍作七十以

Answer is 23

3	70
5	21
7	15

2 x 70 = 140
3 x 21 = 63
2 x 15 = 30
Add up to get 233.
Modify by multiples of 105 to get 128, 23, 338, 443, etc.

T'is hard to find one man of seventy out of three.
There are twenty-one branches on five plum blossom trees.
When seven persons meet, it is in the middle of the month.
Discarding one hundred and five, the problem is done.

THEOREM 17 (CHINESE REMAINDER THEOREM).* *A Dedekind domain R possesses the following property:*

(CRT) *Given a finite number of ideals $\mathfrak{a}_i$ and of elements x_i of R $(i = 1, \cdots, n)$, the system of congruences $x \equiv x_i \pmod{\mathfrak{a}_i}$ admits a solution x in R if and only if these congruences are pairwise compatible, that is, if and only if we have $x_i \equiv x_j \pmod{\mathfrak{a}_i + \mathfrak{a}_j}$ for $i \neq j$.*

PROOF. The property (CRT) is related to the fact that in the set of ideals of a Dedekind domain R, each of the operations $\cap$ and $+$ is *distributive* with respect to the other; that is, that given three ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{b}'$ in R, we have:

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{b}') = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{b}')$$

$$\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{b}') = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{b}').$$

* A rule for the solution of simultaneous linear congruences, essentially equivalent with Theorem 17 in the case of the ring J of integers, was found by Chinese calendar makers between the fourth and the seventh centuries A.D. It was used for finding the common periods to several cycles of astronomical phenomena.

Chinese Remainder Theorem

Oscar Zariski & Pierre Samuel, *Commutative Algebra, Vol. I* (1958), Chapter V, p.279.

The North-China Herald.

JOTTINGS ON THE SCIENCE OF THE CHINESE.

ARITHMETIC.

(Continued from No. 113.)

In examining the productions of the Chinese, one finds considerable difficulty in assigning the precise date for the origin of any mathematical process; for on al-

No. 116.—October 16, 1852.

most every point, where we consult a native author, we find references to some still earlier work on the subject. The high veneration with which it has been customary for them to look upon the labours of the ancients, has made them more desirous of elucidating the works of their predecessors, than of seeking fame in an untrodden path; so that some of their most important formulæ have reached the state in which we now find them by an almost innumerable series of increments. One of the most remarkable of these is the 大衍 *Ta-yen* "Great Extension," a rule for the resolution of indeterminate problems. This rule is met with in embryo in Sun Tsze's Arithmetical Classic * under the name of 物不知數 *Wuh puh che soo* "Unknown numerical quantities," where after a general statement in four lines of rhyme, the following question is proposed:—

Given, an unknown number, which when divided by 3, leaves a remainder of 2; when divided by 5, it leaves 3; and when divided by 7, it leaves 2; what is the number? Ans. 23.

This is followed by a special rule for working out the problem, in terms sufficiently concise and eliptical, to elude the comprehension of the casual reader:—

Dividing by 3 with a remainder of 2, set down 140; dividing by 5 with a remainder of 3, set down 63; dividing by 7 with a remainder of 2, set down 30; adding these sums together gives 233, from which subtract 210, and the remainder is the number required.

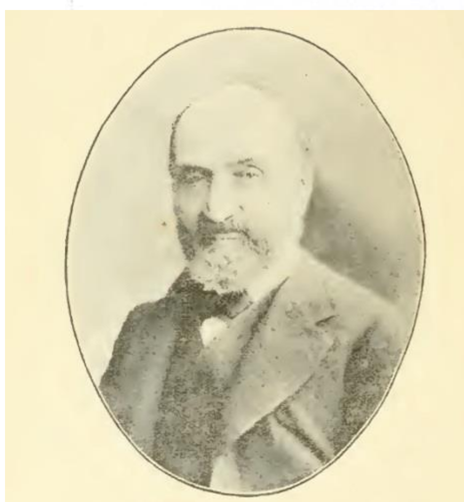
A more general note succeeds:—

For 1 obtained by 3, set down 70; for 1 obtained by 5, set down 21; for 1 obtained by 7, set down 15; when the sum is 106 or above, subtract 105 from it, and the remainder is the number required.

In tracing the course of this process, we find it gradually becoming clearer, till towards the end of the Sung dynasty, when the writings of Tsin Keu-chaou put us in full possession of the principle, and enable us to unravel the meaning of the above mysterious assemblage of numerals. Applying the principles of the *Ta-yen* as there laid down;—Multiplying together the three divisors 3, 5, and 7, gives 105 for the 衍母 *Yen-moo* "Extension parent." Divide this by the 定母 *Ting-moo* "Fixed parent" 7, the quotient 15 is the 衍數 *Yen-soo* "Extension number." Divide this again by 7, and there is an overplus of 1, which is the 乘率 *Ching-suh* "Multiplying term;" by which, multiply the Extension number 15, and the product 15 is the 用數 *Yung soo* "Use number," or as it is given above,—*for* 1 *obtained by* 7, *set down* 15. Divide the Extension parent 105 by the Fixed parent 5, and the quotient 21 is the Extension number. Divide this again by 5, and the overplus 1 is the Multiplying term. Multiply the Extension number 21 by this, and the product 21 is the Use number; which is given above, —*for* 1 *obtained by* 5, *set down* 21. Divide the Extension parent 105 by the Fixed parent 3, and the quotient 35 is the Extension number. Divide this again by 3, and there is a 奇 *Ke*, "Remainder" of 2. This Remainder being more than unity is then submitted to a subsidiary process termed 求一 *Keu Yih* "Finding unity," which is the alternate division of the Extension parent and Remainder by each other, till the remainder is reduced to 1; the result in the present instance is 2 which is the Multiplying term; by which multiply the Extension number, and the product 70 is the Use number; which is the meaning of the sentence,—*for* 1 *obtained by* 3, *set down* 70. Having thus obtained the several Use numbers, multiply the corresponding original remainders by these;—

$70\times2=140$; $21\times3=63$; $15\times2=30$;

add these three numbers together as stated in the rule, and the sum is 233; from which subtract as many times the parent number 105 as it will admit, which making 210, the remainder is 23, the number required.



Alexander Wylie.

Alexander Wylie (1815-1887)

A. Wylie, *Jottings on the Science of the Chinese. Arithmetic, North China Herald*, 108-121, 1852.

The general principles of the *Ta-yen* are probably given in their simplest form, in the above rudimentary problem of Sun Tsze; Subsequent authors enlarging on the idea, applied it with much effect to that complex system of cycles and epicycles which form such a prominent feature in the middle-age astronomy of the Chinese. The reputed originator of this theory as applied to astronomy is the priest Yih Hing who had scarcely finished the rough draft of his work 大衍曆書 *Ta-yen leih shóo,* when he died A.D. 717.

But it is in the "Nine sections of the art of numbers" by Tsin Keu chaou that we have the most full and explicit details on this subject. "Here we have the various applications of this theory worked out at great length; the first problem being to find a solution of a passage in the Yih King treating of the origin of the divining numbers:—

Qu. In the Yih King it is said,—'The Great Extension number is 50, and the Use number is

* Native writers are divided in opinion as to the time when Sun Tsze lived; some consider him the same as Sun Woo-tsze, a military officer during the Heptarchy about B.C. 220. The more probable opinion however, is that he lived towards the end of the Han or during the Wei dynasty in the third century of the Christian era.

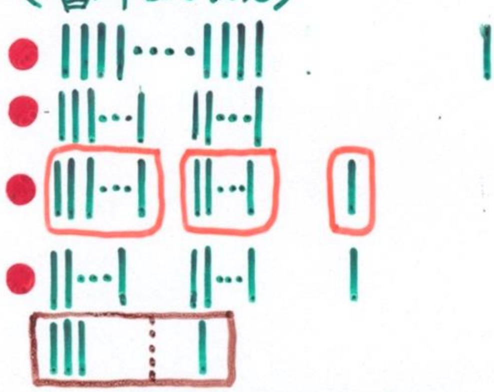
***Da Yan* 大衍 (Great Extension) Art of Searching for Unity**

Qin Jiu-shao 秦九韶, *Shushu Jiuzhang* 數書九章 (Mathematical Treatise in Nine Sections), 1247

Shushu Jiuzhang 數書九章，Book I, Problem 1

In the Yih King [Yi Jing] it is said, "The Great Extension number is 50, and the Use number is 49." Again it is said, "It is divided into 2 [parts], to represent the spheres; 1 is suspended to represent the 3 powers; they are drawn out by 4, to represent the 4 seasons; three changes complete a symbol, and eighteen changes perfect the diagrams." What is the rule for the Extension and what are the several numbers? ……

Fortune telling by combination of shi grass (蓍卦發微)



- Repeat the procedure to the remaining (40 or 44) rods [4 times]

$$50-1=49$$

$$A+B=49$$

$$\boxed{A}+\boxed{B-1}+\boxed{1}=49$$

$$A=4a+\alpha$$

$$B-1=4b+\beta$$

$$\alpha,\beta\in\{1,2,3,4\}$$

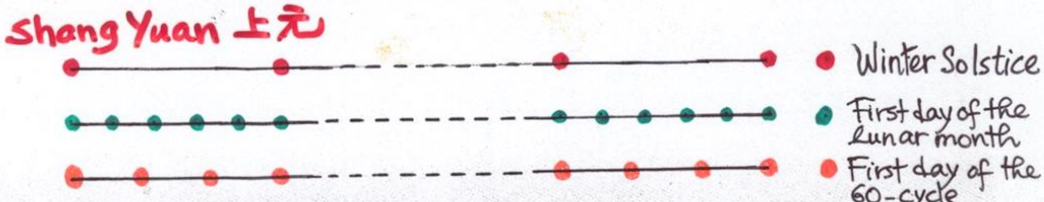
Note that $\alpha+\beta=4$ or 8

about fortune telling

Shushu Jiuzhang 數書九章, Book I, Problem 2

Let the solar year be equal to $365\frac{1}{4}$ days, the moon's revolution, $29\frac{499}{940}$ days, and the *Kea-tsze*, 60 days. Suppose in the year A.D. 1246, the 53d day of the *Kea-tsze* or sexagenary cycle of days is the 1st of the 11th month; the 57th day of the *Kea-tsze* is the Winter solstice or 1st day of the solar year; and the 1st day of the *Kea-tsze* is the 9th day of the month. Required the time between two conjunctions of the commencement of these three cycles; also, the time that has already elapsed, and how much has yet to run. *Ans.* The time between two conjunctions, 18,240 years: 225,600 months: 6,662,160 days: number of years already past 9,163: number of years unexpired, 9,077.

古曆會積



N days in a Shang Yuan period

$N \equiv 0 \pmod{365\frac{1}{4}}$ $\quad x = 4 \times 940 \times N$

$N \equiv 0 \pmod{29\frac{499}{940}}$ $\quad x \equiv 0 \pmod{1373340}$

$N \equiv 0 \pmod{60}$ $\quad x \equiv 0 \pmod{111036}$

$x \equiv 0 \pmod{225600}$

$x = 2087476800,\ N = 555180$

$N \equiv 4 \pmod{365\frac{1}{4}}$

$N \equiv 8 \pmod{29\frac{499}{940}}$

$N \equiv 0 \pmod{60}$

Time elapsed since Shang Yuan $=$ N days

about calendrical reckoning

Shushu Jiuzhang 數書九章, Book II, Problem 5

The 9th problem is as follows :—

A report being raised that 3 rice bins each containing the same amount, have been robbed, the original quantity is not known, but it is found that in the left hand one, there is still 1 *ho* left; in the middle one, there is 1 *shing* 4 *ho* left; in the right hand one, there is 1 *ho* remaining; the thieves being caught, A confesses that he took a horse-ladle at night and filled it several times out of the left hand bin, putting the contents in a bag; B confesses having hastily taken a wooden shoe several times full, out of the middle bin; C says he took a bowl and filled it successively out of the right-hand bin. Examining the three vessels, the horse ladle is found to contain 1 *shing* 9 *ho*, the wooden shoe, 1 *shing* 7 *ho*, and the bowl, 1 *shing* 2 *ho*. What is the amount of rice lost, and how much did each take? Ans. Lost, 9 *shih* 5 *tow* 6 *shing* 3 *ho*. Stolen by A, 3 *shih* 1 *tow* 9 *shing* 2 *ho*; B, 3 *shih* 1 *tow* 7 *shing* 9 *ho*; C, 3 *shih* 1 *tow* 9 *shing* 2 *ho*.

9563

$$x \equiv 1 \pmod{19}$$
$$x \equiv 14 \pmod{17}$$
$$x \equiv 1 \pmod{12}$$

$19 \times 17 \times 12 = 3876$

$3876/19 = 204$

$3876/17 = 228$

$3876/12 = 323$

$204k \equiv 1 \pmod{19}$	$k = 15$
$228k \equiv 1 \pmod{17}$	$k = 5$
$323k \equiv 1 \pmod{12}$	$k = 11$

$$x = 1 \times 204 \times 15 + 14 \times 228 \times 5 + 1 \times 323 \times 11 - 5 \times 3876 = 3193$$

$3193 - 1 = 3192$, $3193 - 14 = 3179$, $3193 - 1 = 3192$

$3192 + 3179 + 3192 = 9563$

about three thievies stealing rice

又有圓田周一百八十一步徑六十步三分步

之一臣淳風等謹按周三徑一周一百八十一步徑六十步三分步之一依密率徑五十

七步二十二分步之十三問爲田幾何

荅曰十一畝九十步十二分步之一

此於徽術當爲田十畝二百八步三百一十四分步之一百一十三臣

淳風等謹依密率爲田十畝二百五步八十八分步之八十七

術曰半周半徑相乘得積步按半周爲從半徑爲廣故

廣從相乘爲積步也假令圓徑二尺圓中容六弧之一面與圓徑之半其數均等令

徑率一而弧周率三也又按爲圖以六弧之一面乘一弧半徑二因而六之得十二

弧之羃若又割之次以十二弧之一面乘一弧之羃半徑四因而六之則得二十四弧

之羃割之彌細所失彌少割之又割以至於不可割則與圓周合體而無所失矣

面之外猶有餘徑以面乘徑則羃出弧表若夫弧之細者與圓合體則表無餘徑表

無餘徑則羃不外出矣以一面乘半徑弧而裁之每輒自倍故以半周乘半徑而爲

圓羃此以周徑謂至然之數非周三徑一之率也周三者從其六弧之環耳以推圓

Chapter 1 (Field Measurement)
Problem 32 : A circular field has a perimeter of 181 *bu* and a diameter of 60 and 1/3 *bu*. What is the area?

《九章算術》***Jiuzhang Suanshu***
(Nine Chapters on the Mathematical Art), compiled between 1st century B.C.E. and 1st century C.E.

Somebody told me the area formula of a circle is *not* in the primary school syllabus. Is that so?

No, it is not in the syllabus. It was there before, but has been removed.

Why?

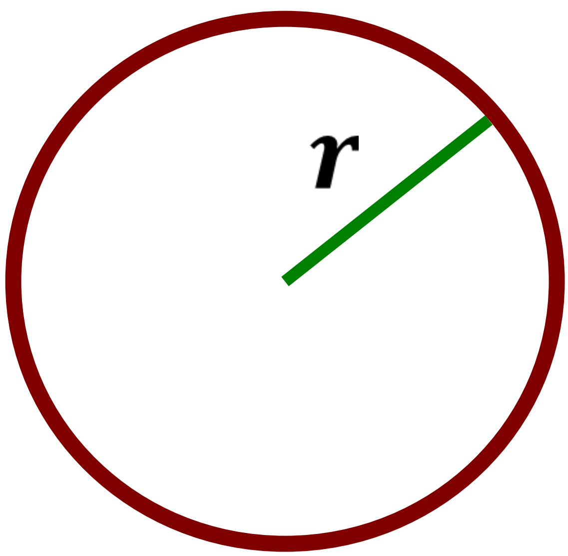
We *cannot prove* the formula at the primary school level.

Can you prove it at the secondary school level?

Yes, we can do it by calculus.

Really?

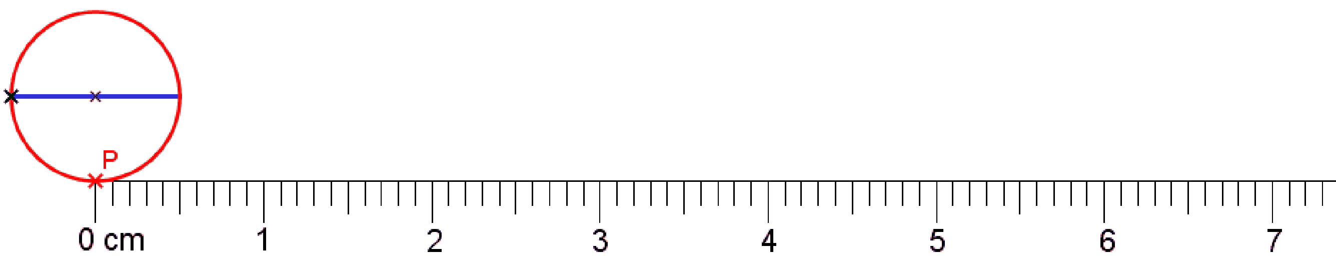
Area of a circle with radius $r = \pi r^2$



(junior primary/senior primary/ junior secondary/senior secondary level ?)

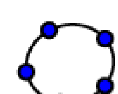
When primary school pupils first encounter the formula for the area of a circle they are convinced of the validity of the formula by a heuristic reasoning.

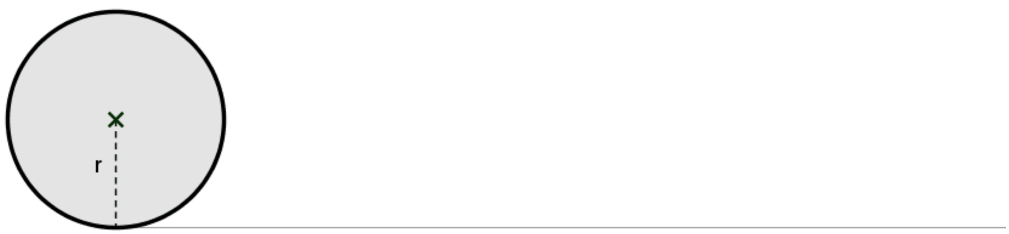
A heuristic reasoning is not a proof, at best a nice argument to help us discover the formula.



One of the various heuristic means to explain the formula for the circumference of a circle

http://ggbtu.be/m8222

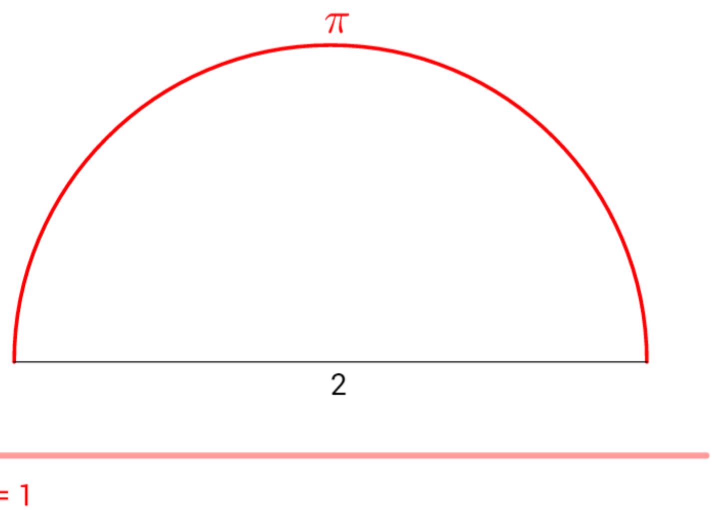




One of the various heuristic means to explain the formula for the area of a circle

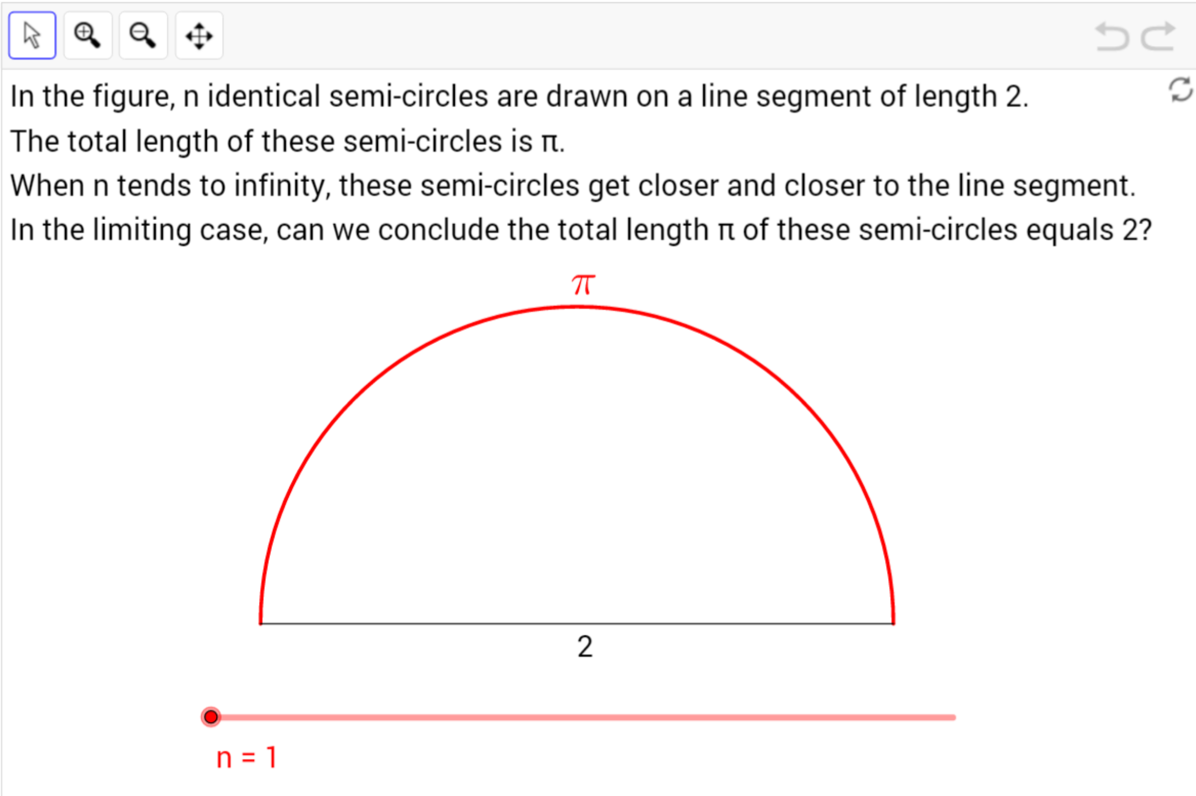
http://ggbtu.be/m279

In the figure, n identical semi-circles are drawn on a line segment of length 2.
The total length of these semi-circles is π.
When n tends to infinity, these semi-circles get closer and closer to the line segment.
In the limiting case, can we conclude the total length π of these semi-circles equals 2?

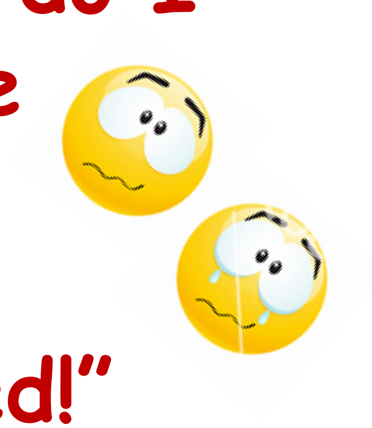


Is $\pi = 2$?

http://ggbtu.be/m1742359



"Good heavens, how do I know when I can use limiting argument and when I cannot? I feel totally baffled!"

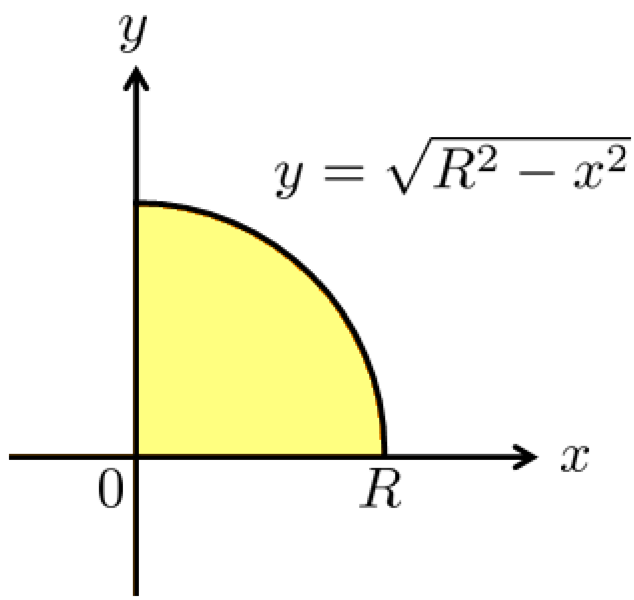


http://ggbtu.be/m1742359

Many cautious teachers will add that a rigorous derivation of the formula rests upon the knowledge of calculus. At a later stage when calculus is taught the area formula would be customarily explained through a certain definite integral.

But does that *really* settle the problem?

Question: How to explain the formula for the area of a circle in a **rigorous** way?



$$\frac{1}{4}A = \int_0^R \sqrt{R^2 - x^2}\,dx$$

$$= \cdots\cdots = \frac{\pi R^2}{4}$$

$$\therefore \boxed{A = \pi R^2}$$

How to compute the integeral

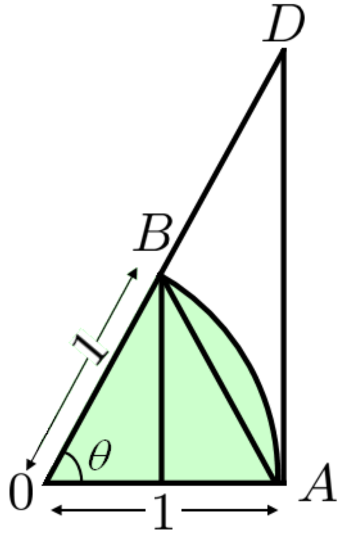
$$\int_0^R \sqrt{R^2 - x^2}dx \ ?$$

How to differentiate the sine/cosine function?

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta, \frac{d}{d\theta}(\cos\theta) = -\sin\theta.$$

Why?

We resort to proving first $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$.

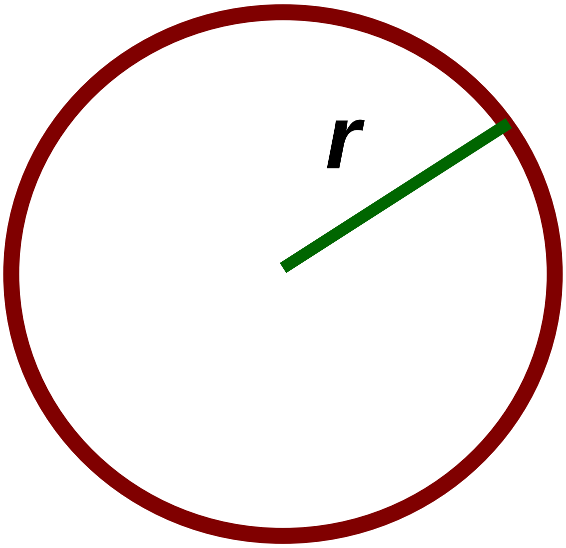


$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\therefore \ \sin\theta < \theta < \tan\theta$$

$$\cdots\cdots$$

Why is the area of the sector OAB equal to $\frac{1}{2}\theta$?



$$A = \pi r^2$$

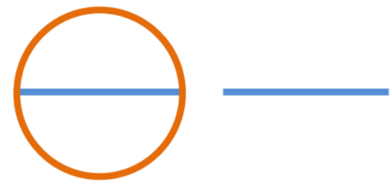
What has been carried out is a circular argument (no pun intended!)

Can we avoid a circular argument?

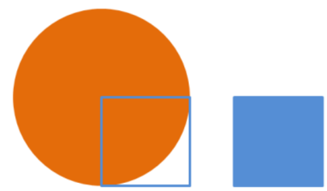
How does the magical constant π enter into the calculation of the **circumference** as well as the calculation of the **area** of a circle?

What happened in history? Can the wisdom of our ancestors enhance our understanding of the problem?

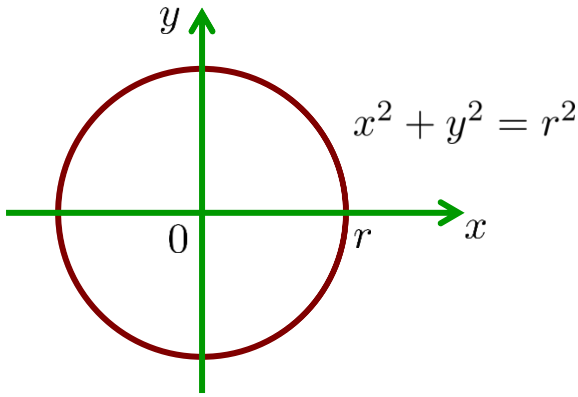
$$\pi = \frac{\text{circumference of circle}}{\text{diameter of circle}}$$



$$\pi = \frac{\text{area of circle}}{\text{area of square on radius of circle}}$$



- ❖ **Which is a "better" definition ?**
- ❖ **Why are they equivalent ?**



$$C = 4\int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx\,.$$
$$A = 4\int_0^r \sqrt{r^2 - x^2}\, dx\,.$$

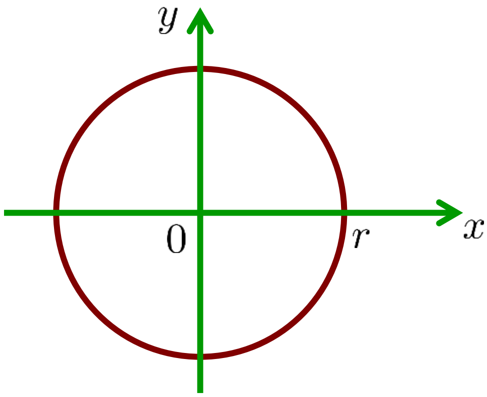
$$\frac{C}{2r} = 2\int_0^r \frac{1}{\sqrt{r^2 - x^2}}\, dx\,.$$
$$\frac{A}{r^2} = 4\int_0^r \frac{\sqrt{r^2 - x^2}}{r^2}\, dx\,.$$

Is $\dfrac{C}{2r} = \dfrac{A}{r^2}$?

That is, is $\displaystyle\int_0^r \frac{r^2}{\sqrt{r^2 - x^2}}\, dx = 2\int_0^r \sqrt{r^2 - x^2}\, dx$?

$$\begin{aligned}
\int_0^r \frac{r^2}{\sqrt{r^2 - x^2}}\, dx &= \int_0^r \frac{r^2 - x^2 + x^2}{\sqrt{r^2 - x^2}}\, dx \\
&= \int_0^r \sqrt{r^2 - x^2}\, dx + \int_0^r \frac{x^2}{\sqrt{r^2 - x^2}}\, dx \\
&= \int_0^r \sqrt{r^2 - x^2}\, dx + \int_r^0 \frac{r^2 - y^2}{y}\left(-\frac{y}{x}\right) dy \\
&= \int_0^r \sqrt{r^2 - x^2}\, dx + \int_0^r \sqrt{r^2 - y^2}\, dy \\
&= 2\int_0^r \sqrt{r^2 - x^2}\, dx\,.
\end{aligned}$$

Bingo ! Clever but contrived "symbol-pushing"! What actually is going on ?



$$
\begin{aligned}
C &= 4\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx\,. \\
A &= 4\int_0^r \sqrt{r^2-x^2}\,dx\,.
\end{aligned}
$$

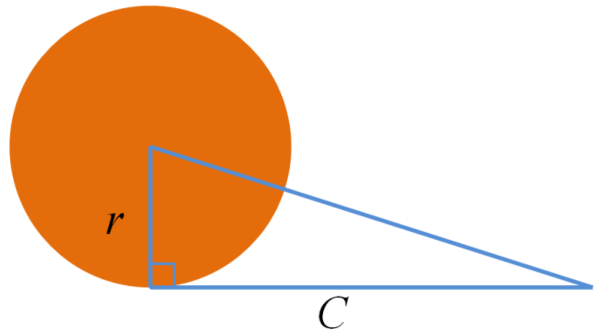
Actually, this shows that

$$\boxed{\boldsymbol{A = \frac{1}{2}Cr}}\,.$$

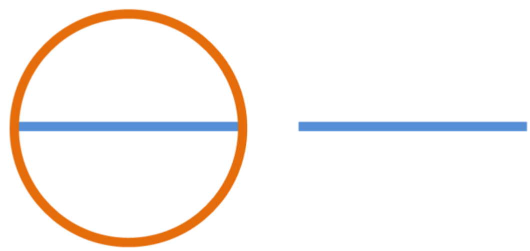
$$
\begin{aligned}
\frac{1}{2}Cr &= 2\int_0^r \frac{r^2}{\sqrt{r^2-x^2}}\,dx \\
&= 2\int_0^r \frac{r^2-x^2+x^2}{\sqrt{r^2-x^2}}\,dx \\
&= \cdots\cdots \\
&= 4\int_0^r \sqrt{r^2-x^2}\,dx = A\,.
\end{aligned}
$$

Now, the formula acquires a nice geometric meaning. It is a great discovery in the ancient world, both in the East and in the West.

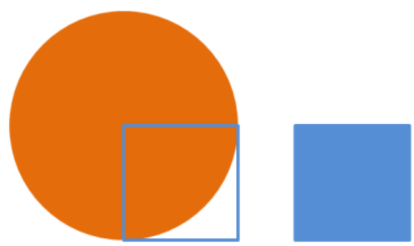
$$\boxed{A = \frac{1}{2}Cr}$$



$$\pi_1 = \frac{C}{2r}$$



$$\pi_2 = \frac{A}{r^2}$$



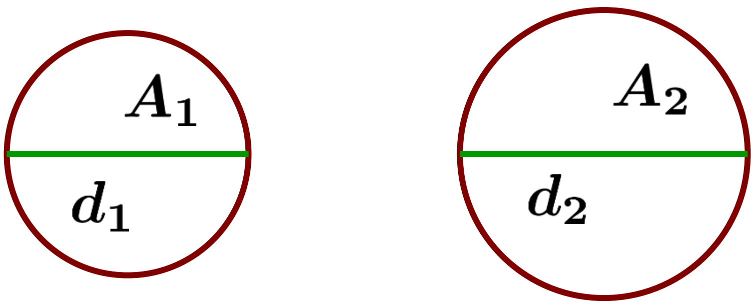
$$\therefore \quad \pi_1 = \frac{C}{2r} = \frac{2A}{r} \times \frac{1}{2r}$$

$$= \frac{A}{r^2}$$

$$= \pi_2$$

Proposition 2, Book XII of Euclid's *Elements*

Circles are to one another as the squares on their diameters.



$$A_1 : A_2 = {d_1}^2 : {d_2}^2$$

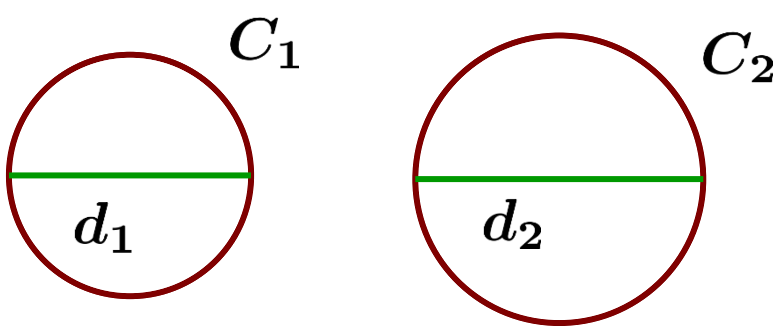
$$\boxed{A = kd^2}$$

A = area
d = diameter

In fact $k = \dfrac{\pi}{4}$,

that is $A = \pi r^2$, r = radius.

Circumferences are to one another as their diameters.



$$C_1 : C_2 = d_1 : d_2$$

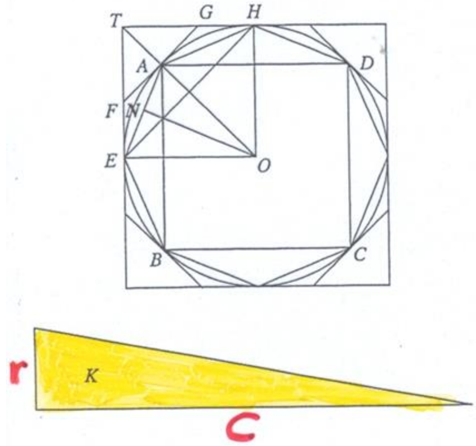
$$\boxed{C = \pi d}$$

C **= circumference**
d **= diameter**

Unlike the area analog, such a theorem in this explicitly and clearly stated form was not documented anywhere in the ancient literature. Since 1994 I have asked many people about this, but found an answer only recently in a 2013 preprint of David Richeson (arXiv: 1303.0904v2). I am glad to see that it agrees with what I have been thinking for some time.

David Richeson, Circular reasoning: who first proved that *C* divided by *d* is a constant?

***The College Mathematics Journal*, 46 (3) (2015), 162-171.**



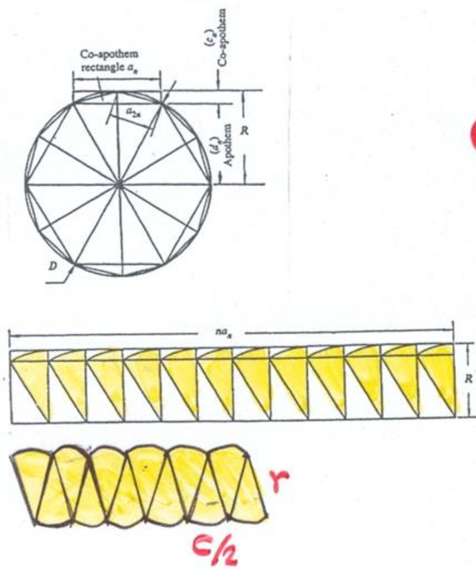
Archimedes

Measurement of a Circle

(3rd century B.C.)

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

$A=\frac{1}{2}Cr$



Liu Hui [劉徽]

Commentary on Jiuzhang Suanshu

[《九章算術》注]

(3rd Century)

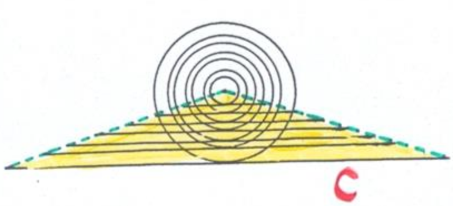
半周半徑相乘得積步

$A=\frac{1}{2}Cr$

Abraham bar Hiyya ha-Nasi

Treatise on Mensuration

(12th century)



The area of any circle is equal to an isosceles triangle with height equal to the radius and with base equal to the circumference of the circle.

$A=\frac{1}{2}Cr$

又有圓田周一百八十一步徑六十步三分步之一臣淳風等謹按周三徑一周一百八十一步徑六十步三分步之一依密率徑五十七步二十二分步之十三問爲田幾何

荅曰十一畝九十步十二分步之一此於徽術當爲田十畝二百八步三百一十四分步之一百一十三淳風等謹依密率爲田十畝二百五步八十八分步之八十七

術曰半周半徑相乘得積步按半周爲從半徑爲廣故廣從相乘爲積步也假令圓徑二尺圓中容六弧之一面與圓徑之半其數均等令徑率一而弧周率三也又按爲圖以六弧之一面乘一弧半徑二因而六之得十二弧之冪若又割之次以十二弧之一面乘一弧之半徑四因而六之則得二十四弧之冪割之彌細所失彌少割之又割以至於不可割則與圓周合體而無所失矣面之外猶有餘徑以面乘徑則冪出弧表若夫弧之細者與圓合體則表無餘徑表無餘徑則冪不外出矣以一面乘半徑弧而裁之每輒自倍故以半周乘半徑而爲圓冪此以周徑謂至然之數非周三徑一之率也周三者從其六弧之環耳以推圓

Chapter 1 (Field Measurement)
Problem 32 : A circular field has a perimeter of 181 *bu* and a diameter of 60 and 1/3 *bu*. What is the area?

《九章算術》Jiuzhang Suanshu
(Nine Chapters on the Mathematical Art),
compiled between
1st century B.C.E. and 1st century C.E.

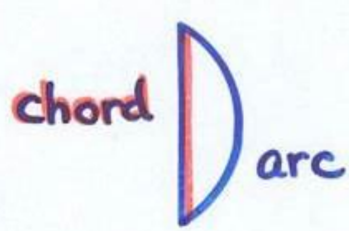
九章算術 卷一 圭

此以周徑謂至然之數非周三徑一之率也周三者從其六觚之環耳以推圓規多少之~~較~~覺（案較原本訛作覺今改正）乃弓之與弦也然世傳此法莫肯精覈學者踵古習其謬失不有明據辯之斯難凡物類形象不圓則方方圓之率誠著于近則雖遠可知也由此言之其用博矣謹按圖驗更造密率恐空設法數昧而難譬故置諸檢括謹詳其記注焉

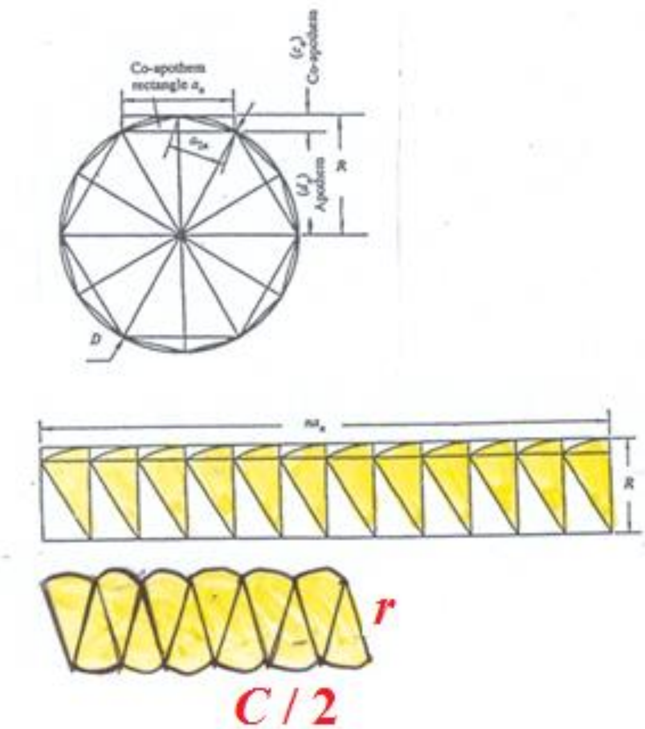
***Jiuzhang Suanshu* 《九章算術》**
(Nine Chapters on the Mathematical Art), ca. 100 B.C.E to 100 C.E.

M.K. Siu, Proof and pedagogy in ancient China: Examples from Liu Hui's Commentary on JIU ZHANG SUAN SHU, *Educational Studies in Mathematics*, 24 (1993), 345-357.

... "In our calculation we use a more accurate value for the ratio of the circumference to the diameter instead of the ratio that the circumference is 3 to the diameter's 1. The latter ratio is only that of the perimeter of the inscribed regular 6-gon to the diameter. Comparing arc with the chord, just like the bow with the string, we see that the circumference exceeds the perimeter". This is apparent from Fig. 2. He continued: "However, those who transmit this method of calculation to the next generation never bother to examine it thoroughly but merely repeat what they learned from their predecessors, thus passing on the error. Without a clear explanation and definite justification it is very difficult to separate truth from fallacy."



LIU Hui (劉徽), *Commentary on Jiuzhang Suanshu* (《九章算術》注) (3rd century)



$$A_6 = \frac{3}{2}a_3 r = \frac{1}{2}C_3 r$$

$$A_{12} = 3a_6 r = \frac{1}{2}C_6 r$$

$$A_{24} = 6a_{12} r = \frac{1}{2}C_{12} r$$

etc.

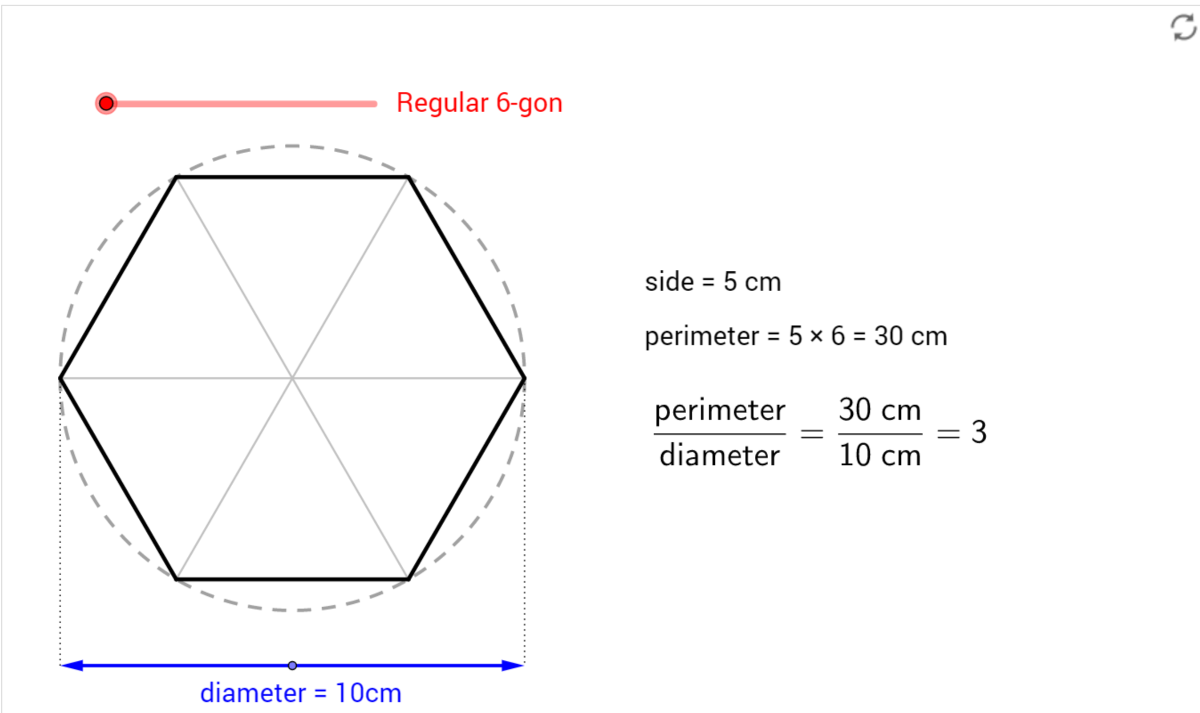
$$A = \frac{1}{2}Cr$$

error estimate:

$$A_{2n} < A < A_{2n} + (A_{2n} - A_n)$$

"Dividing again and again until it cannot be divided further yields a regular polygon coinciding with circle, with no portion whatever left out. (割之又割，以至於不可割，則與圓周合體而無所失矣。)"

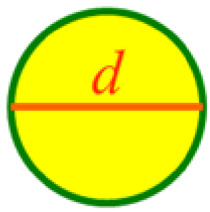
Not rigorous deductive reasoning , but makes sense, leading to the answer through algorithmic means.



http://ggbtu.be/m1743033

$$\boxed{A = \frac{1}{2} C r = \frac{1}{4} C d}$$

$$C = \pi d \qquad A = \frac{\pi}{4} d^2$$



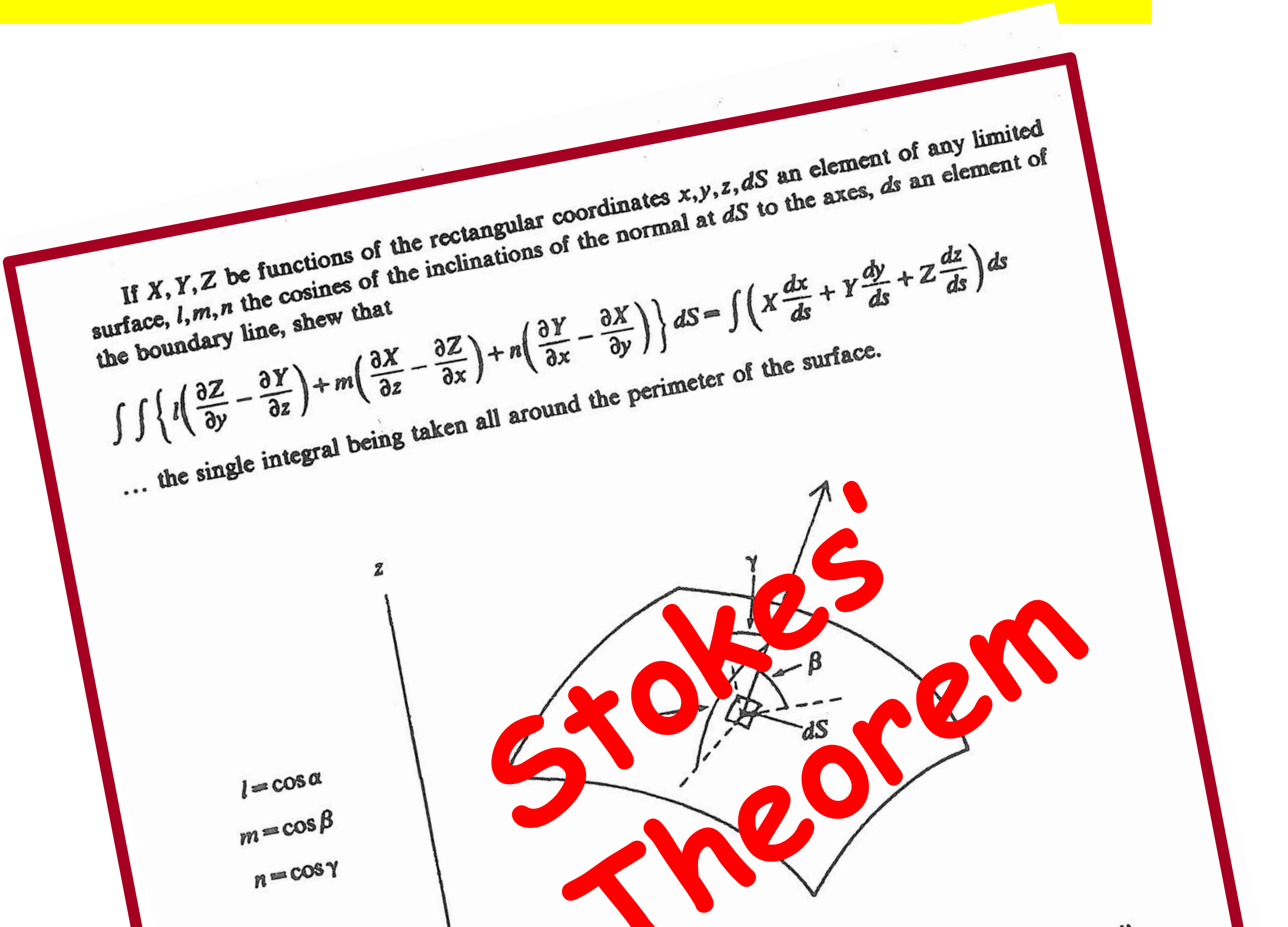
1 – dim. case (boundary)

2 – dim. case (enclosed region)

Fundamental Theorem of Calculus (Stokes' Theorem)

$$\int_{\partial \Omega} \omega = \int_{\Omega} d\omega$$

George Stokes, setting the Smith's Prize Examination at Cambridge University, made this *Question 8* in 1854.



If X, Y, Z be functions of the rectangular coordinates x, y, z, dS an element of any limited surface, l, m, n the cosines of the inclinations of the normal at dS to the axes, ds an element of the boundary line, shew that

$$\iint\left\{l\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)+m\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)+n\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\right\}dS=\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds$$

… the single integral being taken all around the perimeter of the surface.

George Gabriel Stokes (1819-1903)

First wave of transmission of Western learning in China:

late 16th to mid 17th century
Jesuits, Chinese scholar-officials, ministers

… and its wake:

mid 17th to mid 18th century
Jesuits, Emperor Kangxi (康熙), Chinese scholars

Second wave of transmission of Western learning in China:

since mid 19th century
Protestant missionaries, Chinese scholars, Prince Gong [Yixin, 恭親王奕訢] and officials in charge of **"self-strengthening movement (自強運動)"**

The three phases took place within quite different historical contexts and with quite different mentality.

first part of the 17th century [Ming Dynasty]

「欲求超勝，必須會通。」

(In order to surpass we must try to understand and to synthesize.)

first part of the 18th century [Qing Dynasty]

「西學中源」

(Western learning has its origin in Chinese learning.)

latter part of the 19th century [Qing Dynasty]

「師夷長技以制夷」

(Learn the strong techniques of the "[Western] barbarians" in order to control them.)

謝謝香港數理教育學會的邀請，讓我有此機會與大家談談中國古代數學。

謝謝柯志明先生應允作回應嘉賓，與大家分享他的高明識見。

同時，柯志明先生 也協助製作大量*GeoGebra* 程序顯示，以輔助講解。香港大學數學系呂美美女士協助製作圖片為講座添色，謹此一併致謝。