中國古算今譚
—— 從傳統數學
至西學輸入
至現代課堂數學II：

現代中學生/明代徐光啟
初遇上綜合幾何

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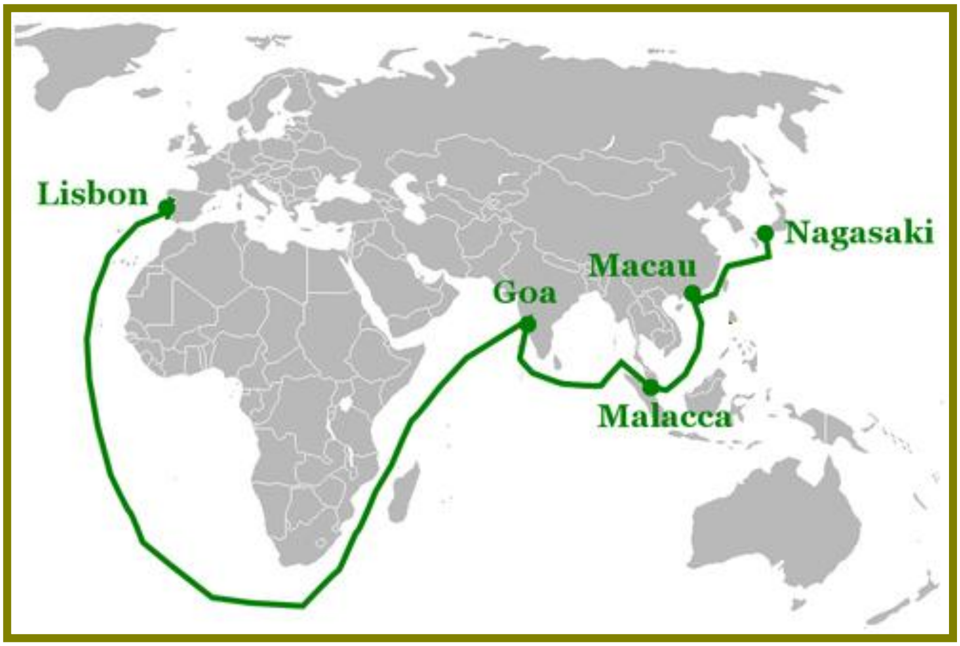
中小學數學課堂上傳授的基本知識和技能，很大部份已經有數百年以至數千年的歷史。從古代至十七世紀的東西方數學典籍當中，記載了相當多的部份。回顧從中國古代至十六世紀的傳統數學，至明末清初西學東漸合流會通，演變成為二十世紀以降在華人教育圈中的現代中小學數學內容。這方面的探討，不只有其數學意義，也富有文化意義，對教與學，都有裨益。

去年五月的講座（「中國古算今譚 — 從傳統數學至西學輸入至現代課堂數學」）可視為這項嘗試的「前傳」，今年五月的講座將會集中討論中學的幾何課程。

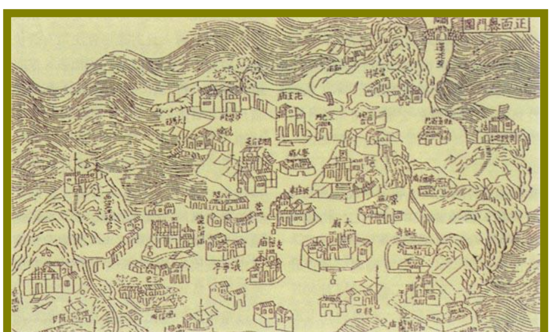
在某種意義上，一位現代中學生初學綜合幾何所碰到的「文化衝擊」，與徐光啟初遇上歐幾里得《原本》的體驗或者有些共通點。固然，今人與四百多年前的古人身處的世界及成長環境有別，不能直接由此及彼作出完全令人信服的推論，但從認知層面而言，嘗試從這個角度切入探討一下亦饒有趣味，甚至可能對教學有所啟迪也說不定。

故事由十五、十六世紀西方的「探索年代」開始。當時歐洲人找到一條通到東方的海路，不同類別的人，因不同的理由來到東方，其中有一批是傳教士。傳教士除了宣傳福音外，還揭開了東西方兩大文明的知識和文化交流重要的一頁。

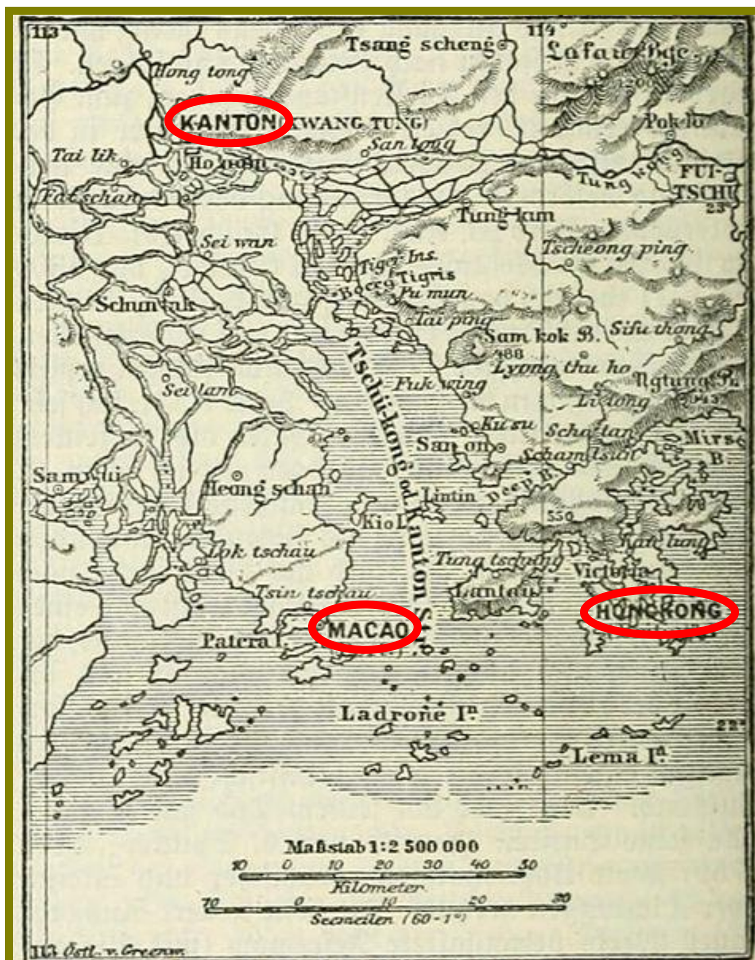
約自1570 年至1650 年這段期間，來到中國宣揚基督教義最突出的傳教士來自創立於 1540 年的耶穌會。本講只述說耶穌會傳教士利瑪竇 (Matteo Ricci , 1552-1610)，而在他把西方學識傳入中國的眾多貢獻中，則只討論他與徐光啟 (1562-1633) 翻譯歐幾里得《原本》的合作經過。



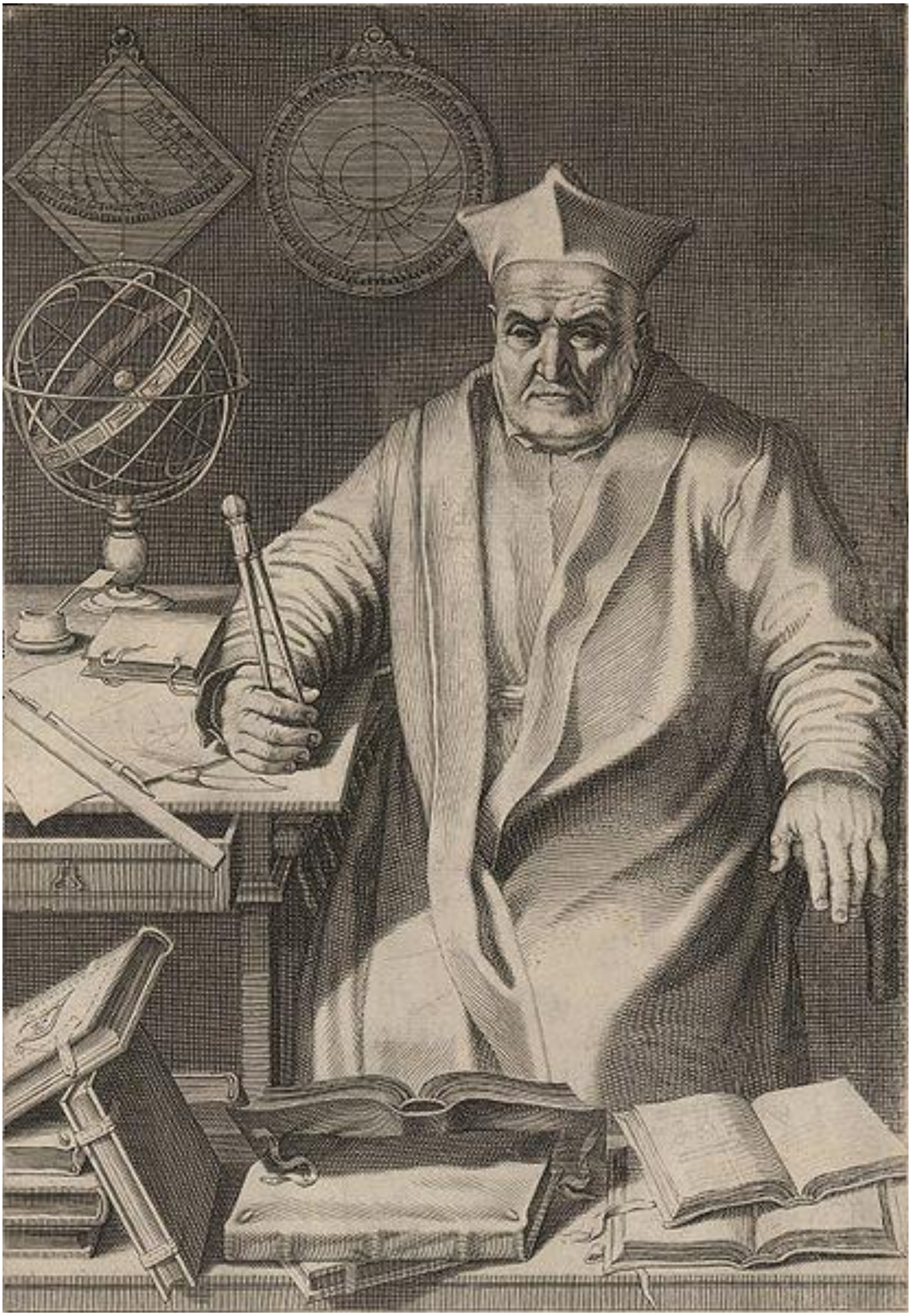
The Portuguese established posts at Goa in 1510 and at Malacca in 1511.
In 1557 the Ming Court gave consent for establishment of an official Portuguese trade post at Macau.



Town map of Macau (17th / 18th century ?)



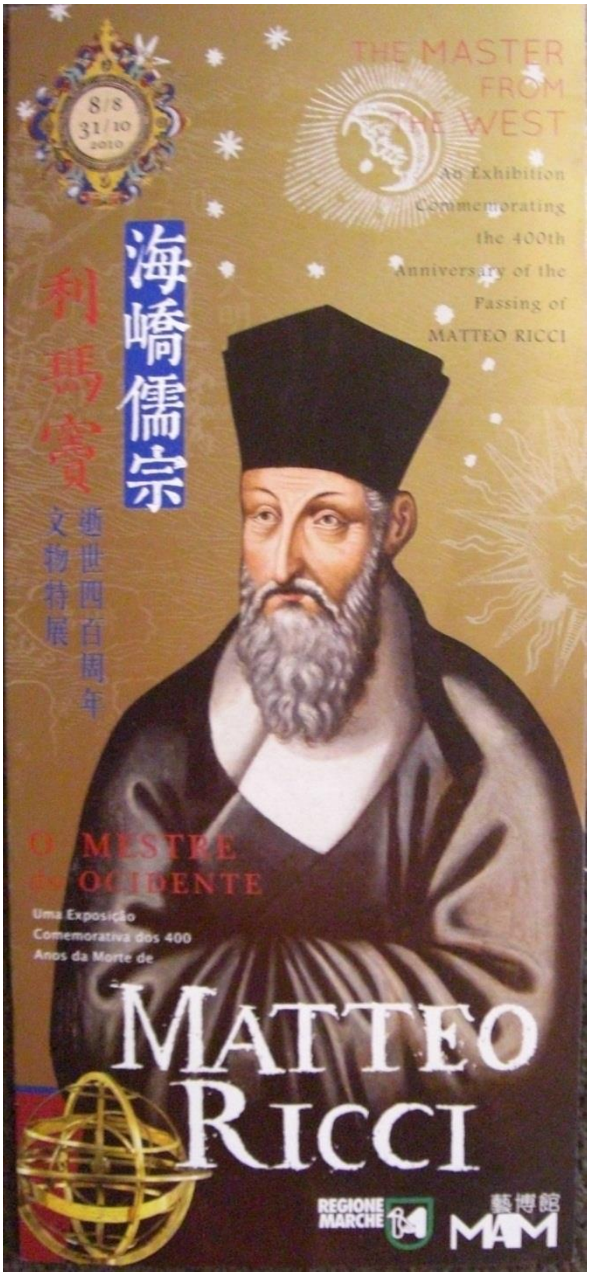
Map of Macau, Hong Kong and Canton [Guangzhou] (late 19th century)

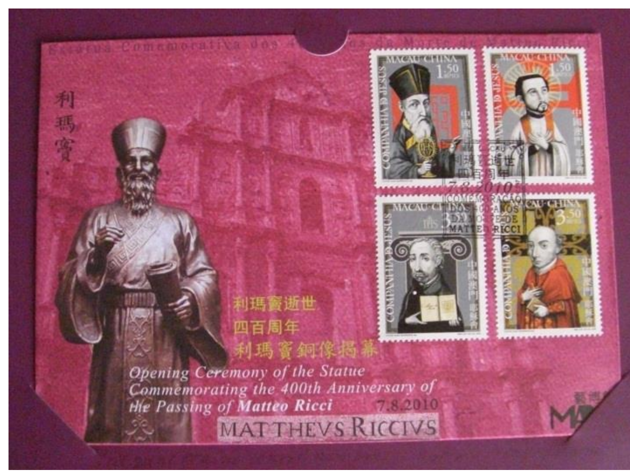


Christopher Clavius (1538-1612)

A statue of Matteo Ricci was erected on August 7, 2010 at the archeological remains of Colégio de Säo Paulo (St. Paul's College) in Macau.

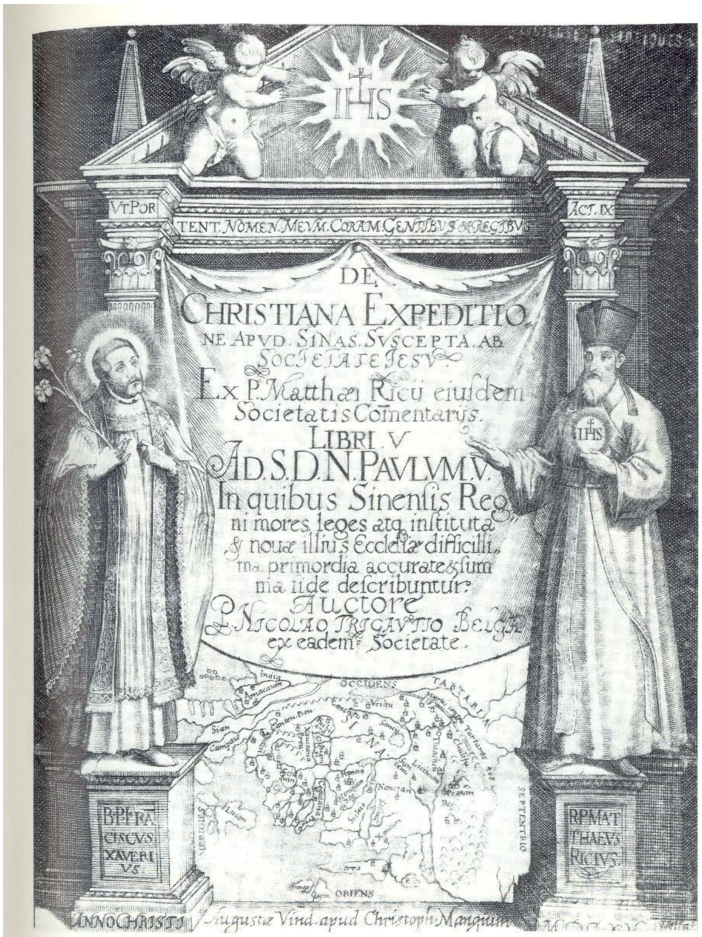
St. Paul's College, founded by the Jesuit Alessandro Valignano (1539-1606) in 1594, was the first western–style university in the Far East.





The Master From the West

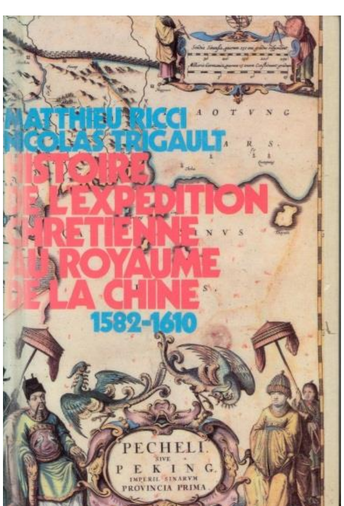
An Exhibition Commemorating the 400th Anniversary of the Passing of Matteo Ricci
Macau Museum of Art
August 7 – October 31, 2010



Matteo Ricci 利瑪竇 (1552-1610)

Nicolas Trigault 金尼閣 (1577-1628)

China in the Sixteenth Century: The Journals of Matthew Ricci, 1583-1610 (transl. L.J. Gallagher, 1942; 1953)



Histoire de l'expedition chretienne au royaume de la Chine, 1582-1610 (transl. G. Bessiere, 1978)

《利瑪竇中國札記》何高濟、王遵仲、李申譯， 1983

"… Whoever may think that ethics, physics and mathematics are not important in the work of the Church, is unacquainted with the taste of the Chinese, who are slow to take a salutary spiritual potion, unless it be seasoned with an intellectual flavouring. … All this, what we have recounted relative to a knowledge of science, served as seed for a future harvest, and also as a foundation for the nascent Church in China…"

China in the Sixteenth Century: The Journals of Matthew Ricci, 1583-1610 **[compiled by Nicolas Trigault and published in 1615; translated from Latin into English by L.J. Gallagher in 1942; 1953]**

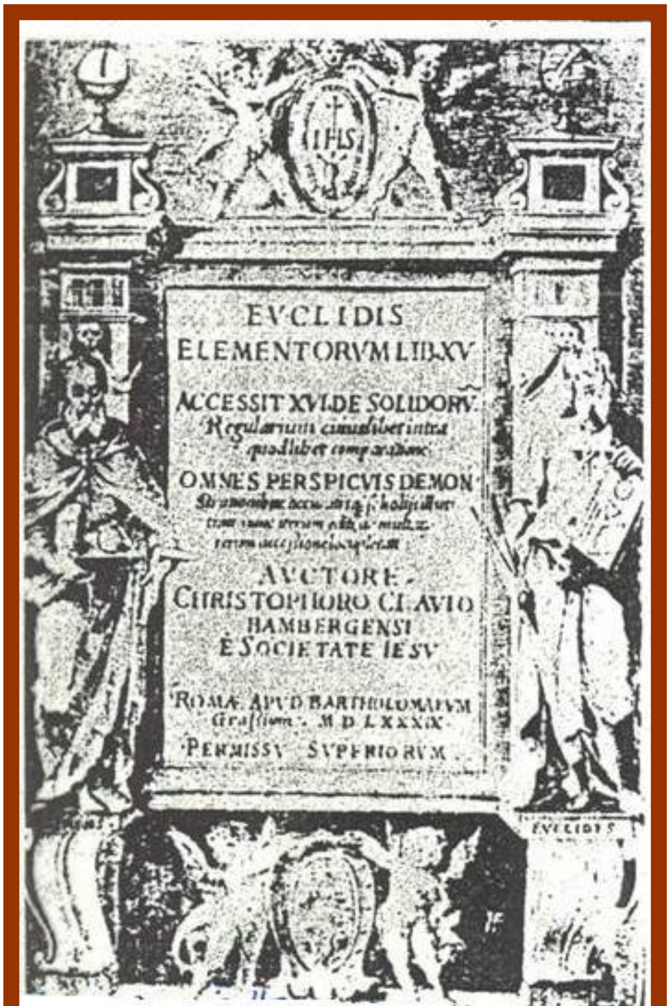
“The result of such a system is that anyone is free to exercise his wildest imagination relative to mathematics, without offering a definite proof of anything. In Euclid, on the contrary, they recognized something different, namely, propositions presented in order and so definitely proven that even the most obstinate could not deny them.”

Is it ?

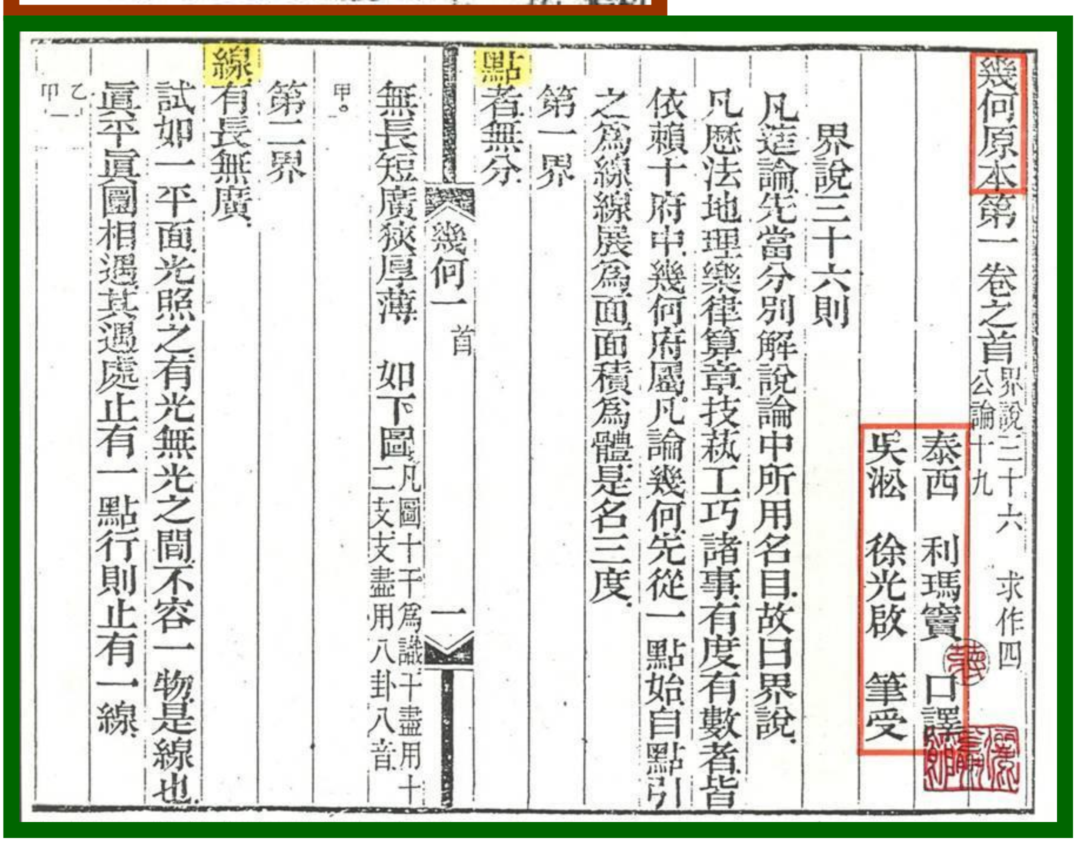
China in the Sixteenth Century: The Journals of Matthew Ricci, 1583-1610 [compiled by Nicolas Trigault and published in 1615; translated from Latin into English by L.J. Gallagher in 1942; 1953]

C. CLAVIUS, *EUCLIDIS ELEMENTORUM LIBRI XV* (1574; 1589)

EUCLID'S *ELEMENTS* (c. 300 B.C.E.)


EVCLIDIS
ELEMENTORVM LIB.XV
ACCESSIT XVI. DE SOLIDORV
Regularium cuiuslibet intra quodlibet comparatione
OMNES PERSPICVIS DEMON
Strationibus accuratisq; scholijs illustrati nunc iterum editi, ac multarum rerum accessione locupletati
AVCTORE
CHRISTOPHORO CLAVIO
BAMBERGENSI
E SOCIETATE IESV
ROMAE APVD BARTHOLOMAEVM
Grassium. M D LXXXIX
PERMISSV SVPERIORVM

Chinese translation by Matteo Ricci and XU Guang-qi (1607)


幾何原本第一卷之首 界說三十六 求作四 公論十九
泰西 利瑪竇 口譯
吳淞 徐光啟 筆受
界說三十六則
凡造論先當分別解說論中所用名目故曰界說
凡歷法地理樂律算章技藝工巧諸事有度有數者皆依賴十府中幾何府屬凡論幾何先從一點始自點引之為線線展為面面積為體是名三度
第一界
點者無分
無長短廣狹厚薄 如下圖 凡圖十干為識干盡用十二支支盡用八卦八音
甲
第二界
線有長無廣
試如一平面光照之有光無光之間不容一物是線也
眞平眞圜相遇其遇處止有一點行則止有一線
乙 甲

幾何原本

"When he [Xu Guang-qi] began to understand the subtlety and solidity of the book, he took such a liking to it that he could not speak of any other subject with his fellow scholars, and he worked day and night to translate it in a clear, firm and elegant style. …… Thus he succeeded in reaching the end of the first six books which are the most necessary and, whilst studying them, he mingled with them other questions in mathematics."

Account by Matteo Ricci (Matteo Ricci, *Opere storiche, I & II,* edited by Father Tacchi Venturi, 1910-1913)

In Henri Bernard (裴化行), Apport scientifique du père Mattieu Ricci à la Chine [translated by Edward Chalmers Werner as *Matteo Ricci's Scientific Contribution to China*, 1935]

“He [Xu Guang-qi] would have wished to continue to the end of the Geometry; but the Father [Matteo Ricci] being desirous of devoting his time to more properly religious matters and to rein him in a bit told him to wait until they had seen from experience how the Chinese scholars received these first books, before translating the others.”

Account by Matteo Ricci (Matteo Ricci, *Opere storiche, I & II,* edited by Father Tacchi Venturi, 1910-1913)

In Henri Bernard (裴化行), Apport scientifique du père Mattieu Ricci à la Chine [translated by Edward Chalmers Werner as *Matteo Ricci's Scientific Contribution to China*, 1935]

「太史意方銳，欲竟之。余曰：止，請先傳此，使同志習之，果以為用也，而後徐計其餘。太史曰：然，是書也，苟為用，竟之何必在我。遂輟譯而梓。」

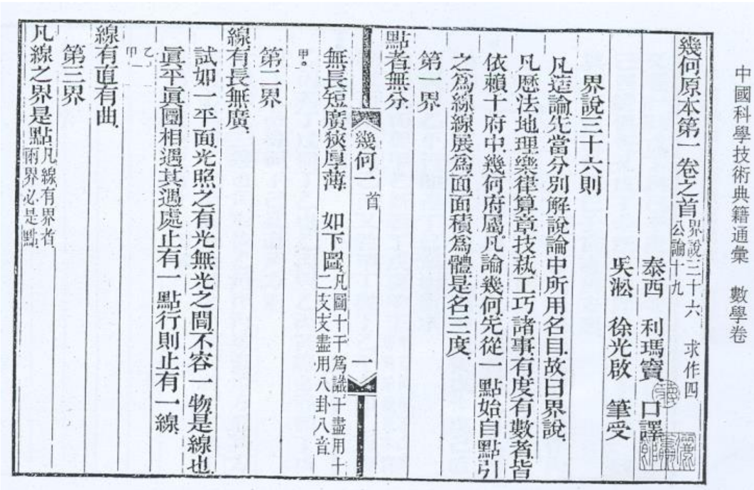
利瑪竇

《幾何原本》序 (1607)

「續成大業，未知何日，未知何人。」

徐光啟‧《幾何原本》修訂版序

(1611)


中國科學技術典籍通彙 數學卷

幾何原本第一卷之首 界說三十六 公論十九 求作四

泰西 利瑪竇 口譯

吳淞 徐光啟 筆受

界說三十六則

凡造論先當分別解說論中所用名目故曰界說

凡歷法地理樂律算章技藝工巧諸事有度有數者皆依賴十府中幾何府屬凡論幾何先從一點始自點引之爲線線展爲面面積爲體是名三度

第一界

點者無分

無長短廣狹厚薄 如下圖 凡圖十干爲識干盡用十二支支盡用八卦八音

甲

第二界

線有長無廣

試如一平面光照之有光無光之間不容一物是線也

眞平眞圜相遇其遇處止有一點行則止有一線

乙 甲

線有直有曲

第三界

凡線之界是點 凡線有界者兩界必是點

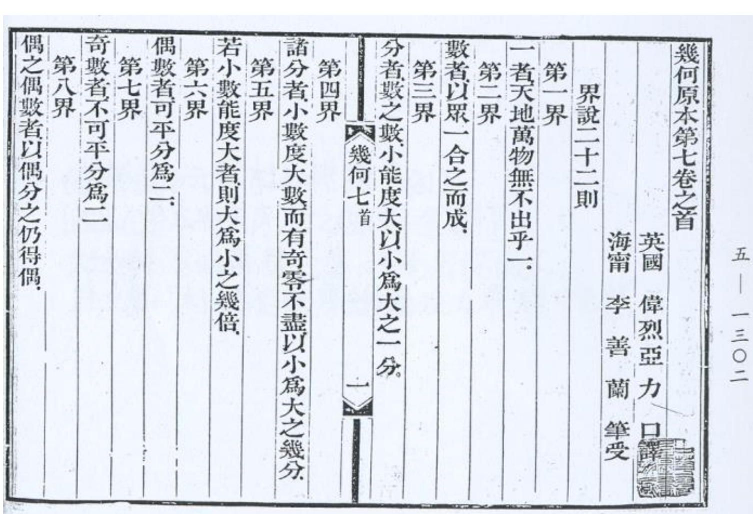
幾何一 首 一

translation by XU Guangqi and Matteo Ricci (1607)

Book I **to** ***Book VI***

(based on Latin compilation by Christopher Clavius, 1574/1589)

250 years !


幾何原本第七卷之首

英國 偉烈亞力 口譯

海甯 李善蘭 筆受

界說二十二則

第一界

一者天地萬物無不出乎一

第二界

數者以眾一合之而成

第三界

分者數之數小能度大以小爲大之一分

幾何七 首 一

第四界

諸分者小數度大數而有奇零不盡以小爲大之幾分

第五界

若小數能度大者則大爲小之幾倍

第六界

偶數者可平分爲二

第七界

奇數者不可平分爲二

第八界

偶之偶數者以偶分之仍得偶

五 — 一三〇二

translation by LI Shanlan and Alexander Wylie (1857)

Book VII **to** ***Book XV***

(based on English translation by Henry Billingsley, 1570)

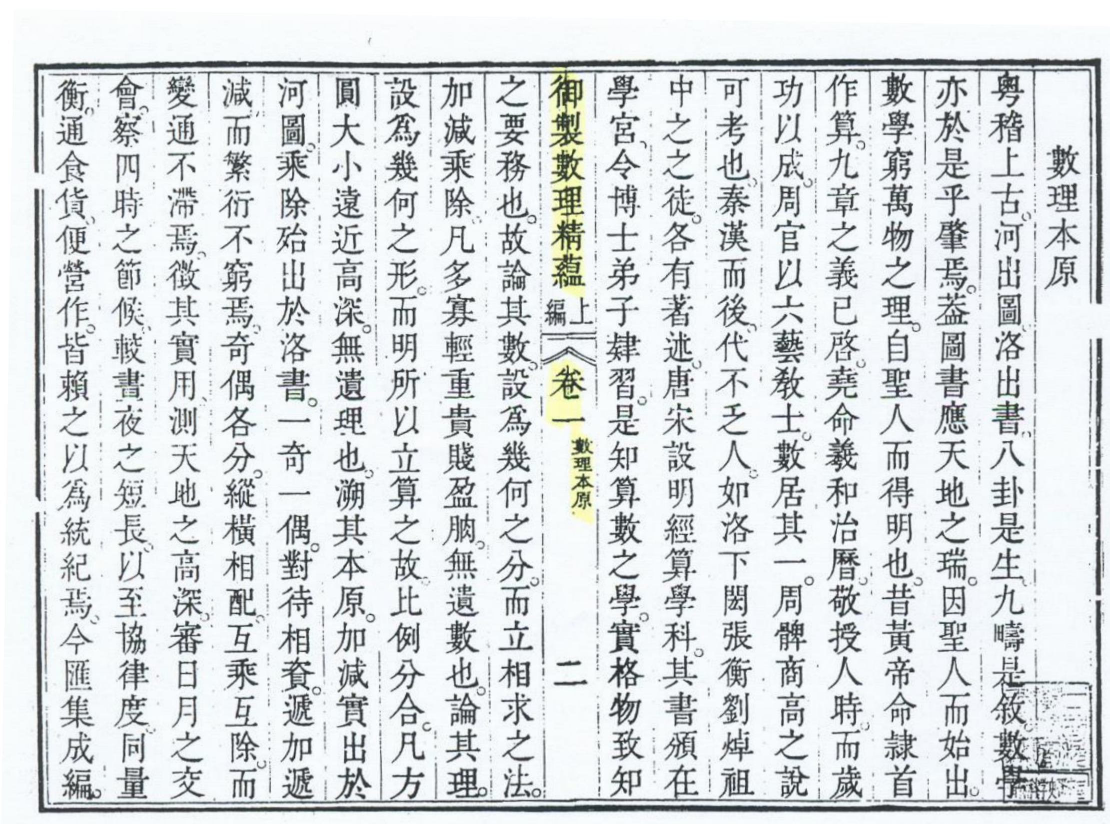
《幾何原本十五卷》金陵足本 (1857/1865) ［偉烈亞力(Alexander Wylie)口譯，李善蘭筆授，於1857年刊行，惜不久即遇上太平兵變及英法聯軍入侵，版燬無傳。遞至曾國藩駐守金陵(即今南京)，李善蘭向曾氏述及此書之重要，曾氏逐出資重印該書，十五卷(前六卷乃明代徐光啟與利瑪竇(Matteo Ricci) 合譯之刻本）於1895年再現中土。］

Courtesy from the Hong Kong University Libraries

Translation of *Book VII* to *Book XV* of *Elements* by LI Shanlan and Alexander Wylie (1857), completing the translation of (fifteen books of) *Elements*.

Compilation of *Lü Li Yuan Yuan* (Origins of Mathematical Harmonics and Astronomy 律曆淵源) commissioned by Emperor Kangxi (project started in 1713, published in 1722/23)

- *Li Xiang Kao Cheng* (Compendium of Observational Computational Astronomy 曆象考成), 42 volumes.
- *Shu Li Jing Yun* (Collected Basic Principles of Mathematics 數理精蘊), 53 volumes.
- *Lü Lü Zheng Yi* (Exact Meaning of Pitchpipes 律呂正義), 5 volumes.



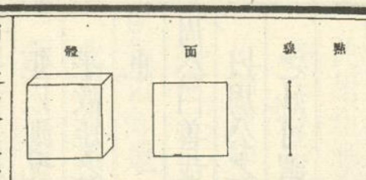
數理本原
粤稽上古河出圖洛出書八卦是生九疇是敘數學
亦於是乎肇焉蓋圖書應天地之瑞因聖人而始出
數學窮萬物之理自聖人而得明也昔黃帝命隸首
作算九章之義已啓堯命羲和治曆敬授人時而歲
功以成周官以六藝教士數居其一周髀商高之說
可考也秦漢而後代不乏人如洛下閎張衡劉焯祖
沖之之徒各有著述唐宋設明經算學科其書頒在
學宮令博士弟子肄習是知算數之學實格物致知
御製數理精蘊 上編 卷一 數理本原 二
之要務也故論其數設為幾何之分而立相求之法
加減乘除凡多寡輕重貴賤盈朒無遺數也論其理
設為幾何之形而明所以立算之故比例分合凡方
圓大小遠近高深無遺理也溯其本原加減實出於
河圖乘除殆出於洛書一奇一偶對待相資遞加遞
減而繁衍不窮焉奇偶各分縱橫相配互乘互除而
變通不滯焉徵其實用測天地之高深審日月之交
會察四時之節候較晝夜之短長以至協律度同量
衡通食貨便營作皆賴之以為統紀焉今匯集成編

《數理精蘊》卷二至卷四［幾何原本］

幾何原本一

第一

凡論數度必始於一點。自點引之而爲線。自線廣之而爲面。自面積之而爲體。是名三大綱。是以有長而無闊者謂之線。有長與闊而無厚者謂之面。長與闊厚俱全者謂之體。惟點無長闊厚薄。其間不能容分。不可以數度。然線之兩端即點。而線面體皆由此生。點雖不入於數。實爲衆數之本。



第二

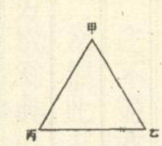
線有直曲兩種。其二線之一端相合。一端漸離。必成一角。二線若俱直者。謂之直線角。一線直一線曲者。謂之不等線角。二線俱曲者。謂之曲線角。

第三

凡角之大小。皆在於角空之寬狹。出角之二線。即如規之兩股。漸漸張去。自然開寬。是以命角不論線之長短。止看角之大小。如丙角兩線雖長。其開股之空狹。遂爲小角。若丁角兩線雖短。其開股之空寬。遂成大角矣。

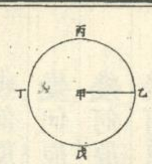
第四

凡命角必用三字爲記。如甲乙丙三角形。指甲角則云乙甲丙角。指乙角則云甲乙丙角。指丙角則云甲丙乙角是也。亦有單舉一字者。則其所舉之一字。即是所指之角也。如單言甲角乙角丙角之類。



第五

凡有一線。以此線之一端爲樞。復以此線之一端爲界。旋轉一周。即成一圜。如甲乙一線。以甲端爲樞。乙端爲界。旋轉復至乙處。即成乙丙丁戊之圜。此圜線謂之圜界。圜界內所積之面度。謂之圜面。



第六

Short, but yet Plain

ELEMENTS

OF

GEOMETRY

AND PLAIN

TRIGOMETRY.

Shewing how by a Brief and Eaſie Method, all that is Neceſſary and Uſeful in *Euclide*, *Archimedes*, *Apollonius* and other Excellent *Geometricians*, both Ancient and Modern, may be Underſtood.

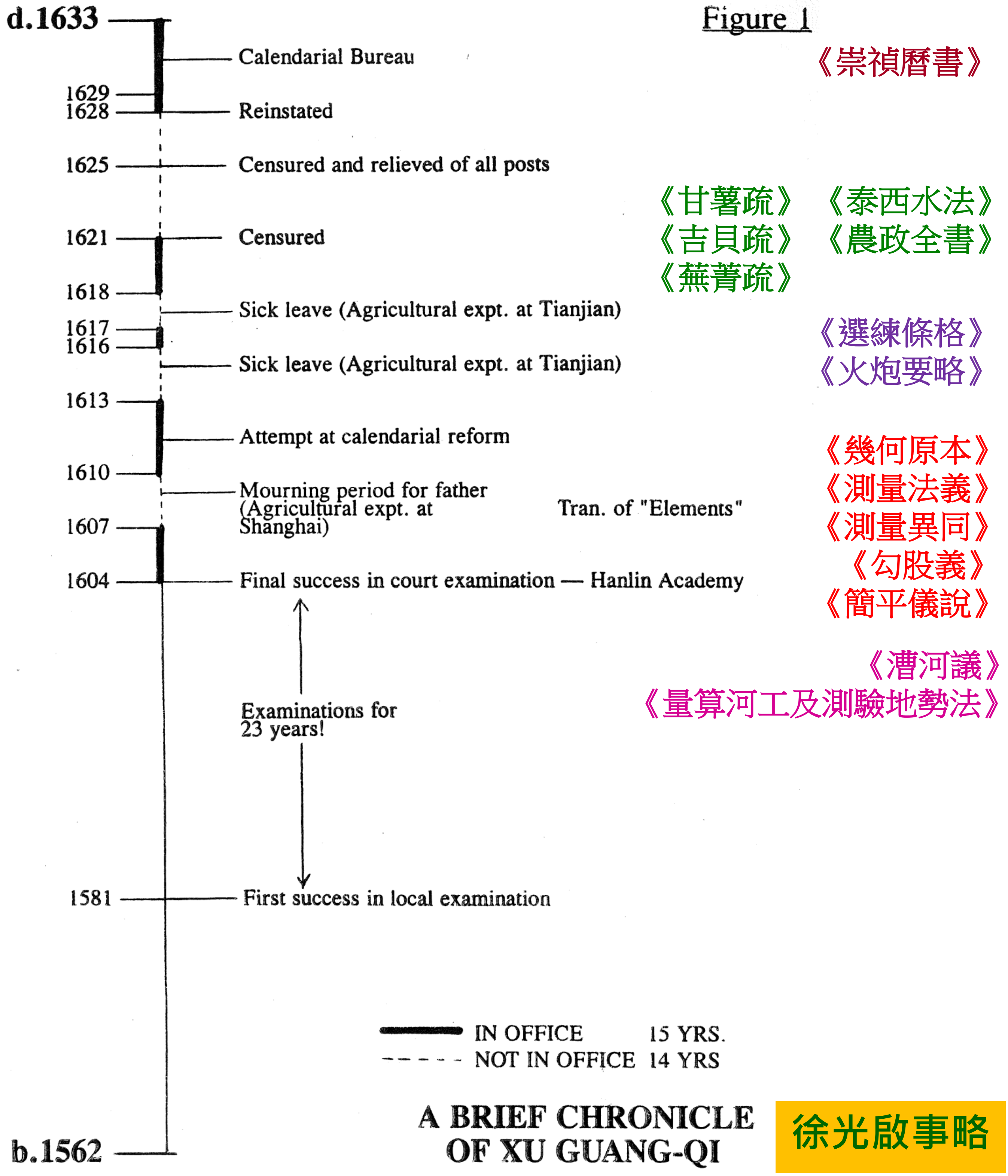
Written in *French*
By *F. Ignatius Gaſton Pardies.*
And now rendred into *Engliſh* from the Fourth and Laſt Edition.
By *John Harris* M. A. and F. R. S.

With many Additions, and Improvements: The whole being Accommodated to the Capacities of Young Beginners.

London, Printed by *J. Matthews*, for *R. Knaplock* at the *Angel*, and *D. Midwinter* and *T. Leigh* at the *Roſe* and *Crown* in St. *Paul's Church-yard*. 1701.

***Elémens de géométrie* by Ignace Gaston Pardies (1636-1673), 1st edition 1671; 6th edition 1705.**

《數理精蘊》卷二至 卷四 [幾何原本]的底本，與利徐二氏翻譯的幾何原本並不相同。



M.K. Siu, Success and failure of XU Guang-qi: Response to the first dissemination of European science in Ming China, *Studies in History of Medicine and Science*, Vol. XIV, Nos. 1-2, New Series (1995/96), 137-179.

「度數旁通十事」：

「其一（天氣），其二（測量），
其三（樂律），其四（軍事），
其五（會計），其六（建築），
其七（機械），其八（輿圖），
其九（醫學），其十（時計）。

右十條於民事似為關切。臣聞之周髀算經云：禹之所以治天下者，句股之所繇生也。蓋凡物有形有質，莫不資於度數故耳。」

徐光啟
XU Guang-qi
(1562-1633)

徐光啟，條議曆法修正歲差疏, 1629

雖然徐光啟強調數學的應用，但他有足夠的視野洞識《原本》本質的特點。

在《幾何原本》刻本序言(1607)他寫道：

「由顯入微，從疑得信，蓋不用為用，眾用所基，真可謂萬象之形囿，百家之學海。」

- 徐光啟如何認識他剛從Clavius 編纂的*Euclidis Elementorum*學到的幾何呢？

- 徐光啟如何理解書中的思想、方法和表達形式，那些都與他熟悉的中國傳統數學很不相同？

西泰子之譯測量諸法也，
十年矣。法而系之義也，
自歲丁未始也。曷待乎？
於時幾何原本之六卷始
卒業矣，至是而後能傳
其義也。是法也、與周
髀九章之句股測望、
異乎？不異也。不異，
何貴焉？亦貴其義也。

徐光啟，題《測量法義》
(1608)

九章算法句股篇中故
有用表、用矩尺測量
數條，與今譯測量法
義相較，其法略同。
其義全闕，學者不能
識其所繇。既具新論，
以考舊文，如視掌矣。

徐光啟，《測量異同》緒言
(1608)

不知其中有理、有義、有法、有數。理不明不能立法，義不辨不能著數。

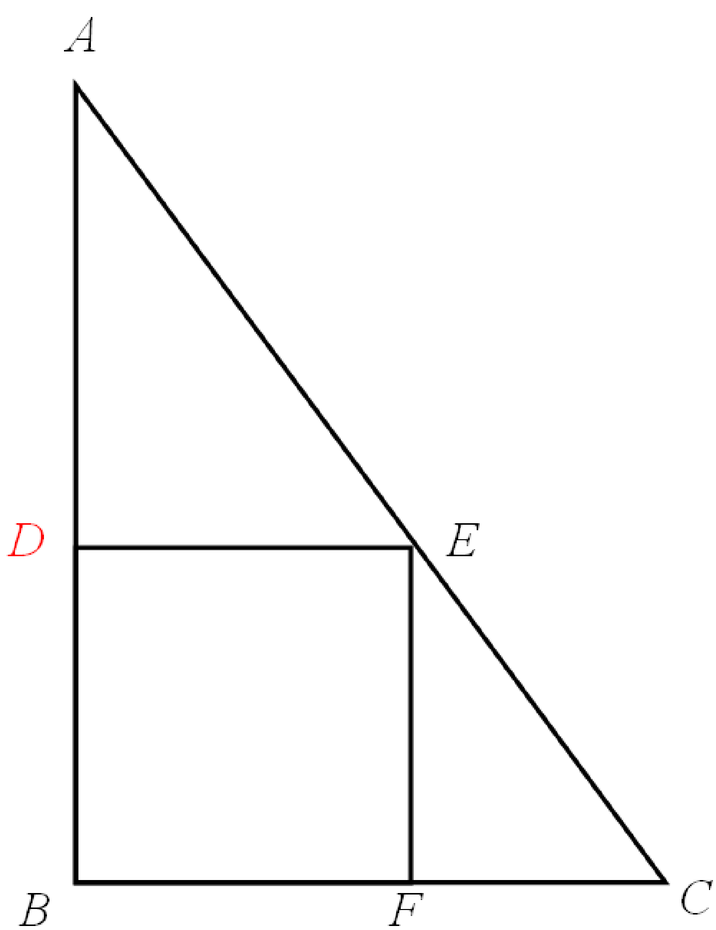
明理辨義，推究頗難；法立數著，遵循甚易。

徐光啟，測候月食奉旨回奏疏 (1629)

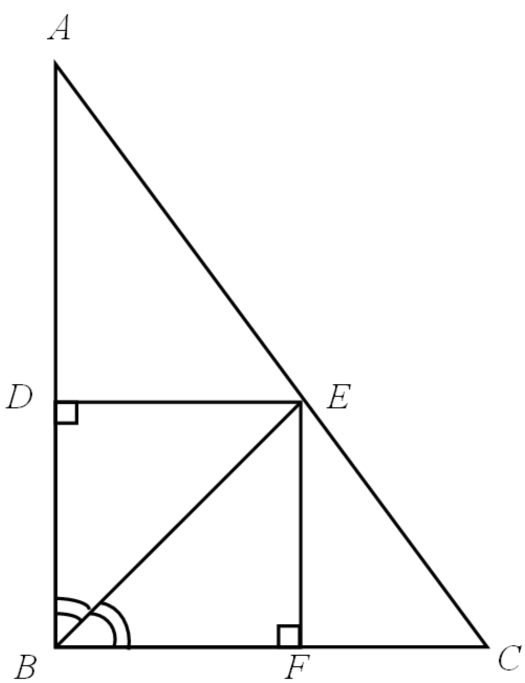
"... but nothing pleased the Chinese as much as the volume on the Elements of Euclid. This perhaps was due to the fact that no people esteem mathematics as highly as the Chinese, despite their method of teaching, in which they propose all kinds of propositions but without demonstrations."

當真?

已知 AC 是直角三角形 ABC 的斜邊，求作三角形的內接正方形 $BDEF$，其中點 D、點 E、點 F 分別在 AB、AC、BC 上。



這道題目並不在歐幾里得《原本》出現。假若《原本》有這一道題目，你猜證明會是怎樣的呢?



BE (E 在 AC 上)平分 $\angle ABC$ [卷一命題九]；

作垂直線ED，EF (D 在 AB 上，F 在 BC 上) [卷一，命題十二]；

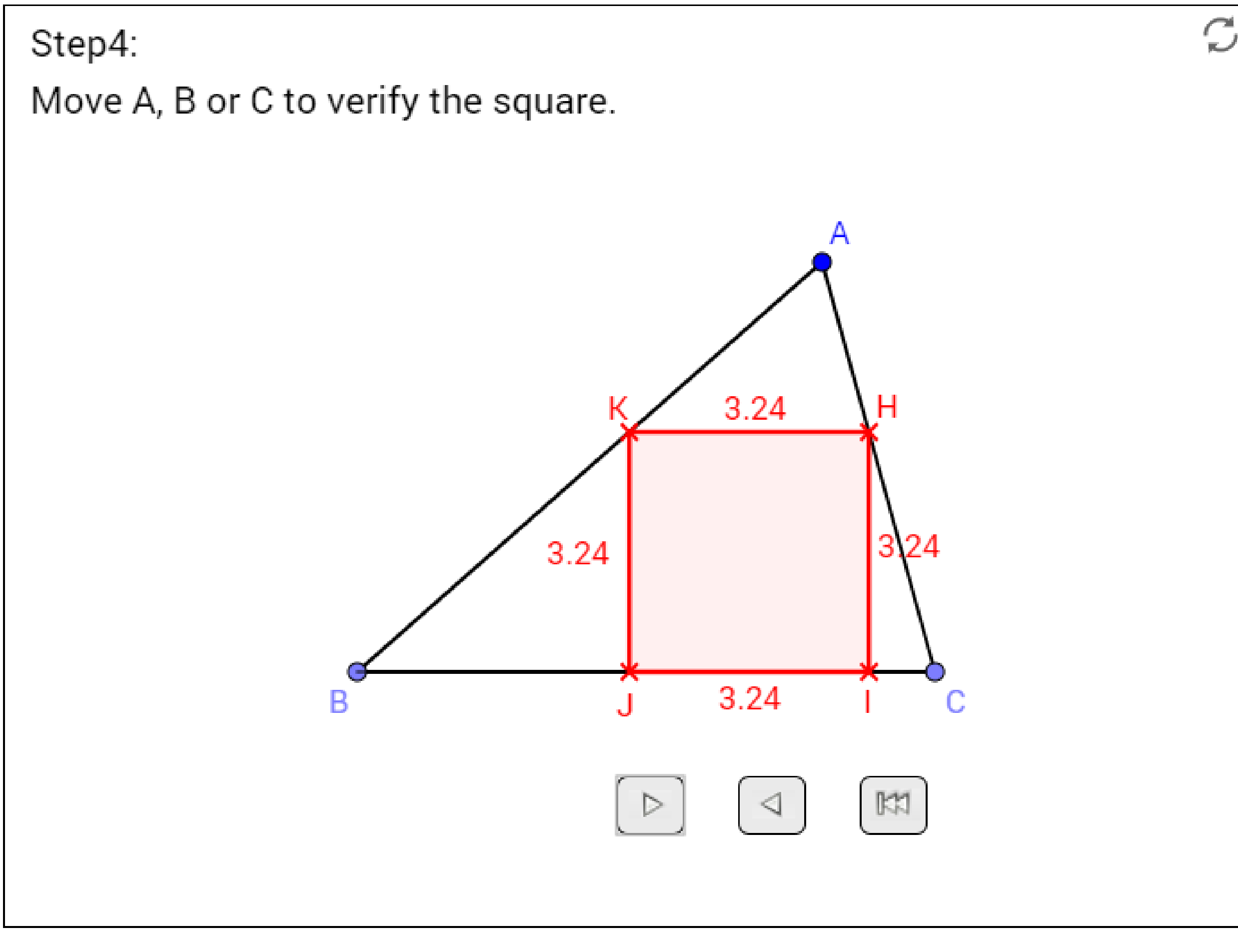
證明 $BDEF$ 是所求的正方形。

習作: 已經知道內接正方形存在，求正方形 DE 的邊長 x。

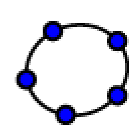
答案: 若 a , b 分別是三角形兩邊(非斜邊)之長 ，則

$$\boxed{x = \frac{ab}{a+b}}$$ 。

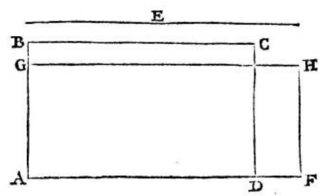
Square Inscribed in a Triangle



http://ggbm.at/6255332



ced into another long ſquare, whoſe length ſhall be equall to the line E.



E. 48.
AB. 24.
BC. 40.
FH. 20.

INcreaſe A D. to F. then ſet the line E. from A. to F. That done, by the 23 Probleme, ſay if E. giue the length B C. what the breadth A B. anſwere A G. for the breadth (of the long ſquare to be made) and A F. or E. is the length, whereof make the long ſquare A G H F. which ſhall be equall to the long ſquare A C. and yet his length A F. equall to the giuen line E. which was required.

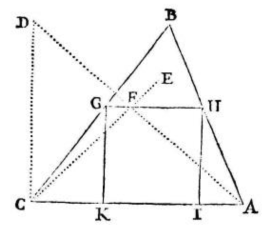
P. 23.

Prob. LXIX.

Within a Triangle, to inſcribe a Square.

Let A B C. be a triangle giuen, wherein a ſquare is to be inſcribed.

Vpon



AB. 169.
AC. 182.
CB. 195.
GH. 84.

VPon the end C. erect a prerpendicular of the length of the perpendicular, from B. vpon the baſe A C. (as C D) then drawe the ſubtendant ſide A D. That done, deuide the right angle A C D. 32. 1.
into two equall parts (by the 8. Probleme) with the line 6. 1.
C E. which cutteth A D. in F. Laſtly, by the point F. 12. 6.
drawe a Parralell to the baſe A C. as G H. whereof make the ſquare G H I K. which ſhall ſtand within the triangle A B C. required.

Prob. LXX.

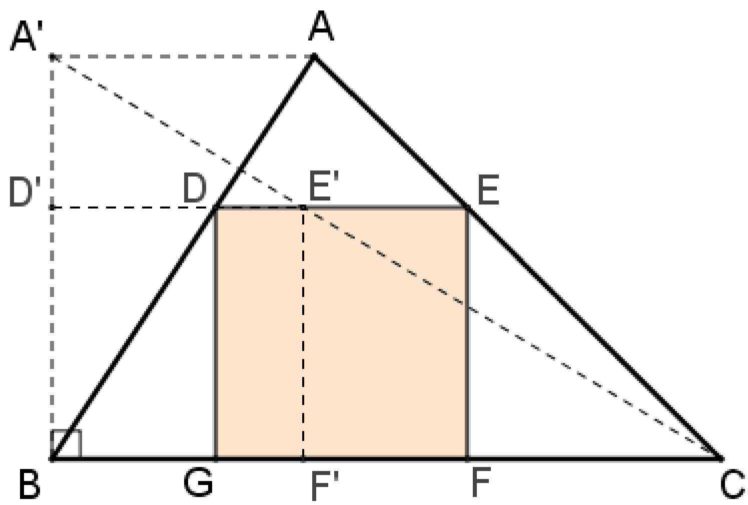
Within a Triangle, to inſcribe a Parralellogram, whoſe ſides ſhall haue proportion together, as two lines giuen.

Let A B C. be a triangle giuen, and let it be required to inſcribe within it a long ſquare, whoſe length ſhall haue proportion to his breadth as the line D. to the line E.

At

Problem 69 in *A geometrical extraction, or, A compendium collection of the chiefest and choisest problems by* John Speidell (1616)

求作三角形 *ABC* 的內接正方形 *DEFG*，其中點 *D*、點 *E* 分別在 *AB*、*AC* 上，點 *F*、點 *G* 在 *BC* 上。

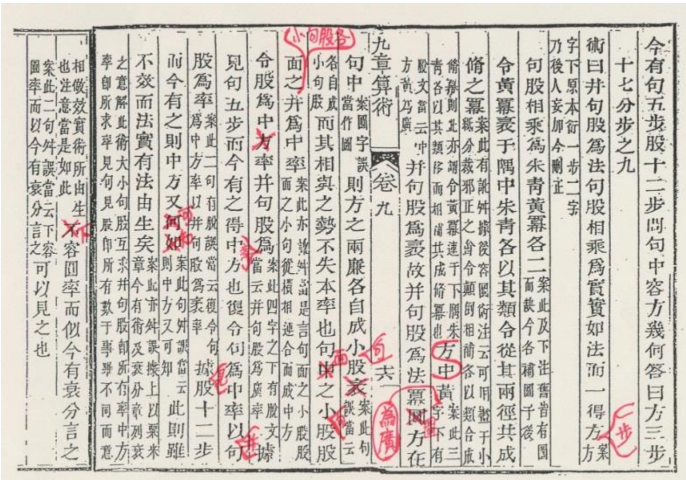


$$\frac{D'D}{A'A}=\frac{BD}{BA}=\frac{CE}{CA}=\frac{E'E}{A'A}$$

$$\therefore D'D = E'E$$

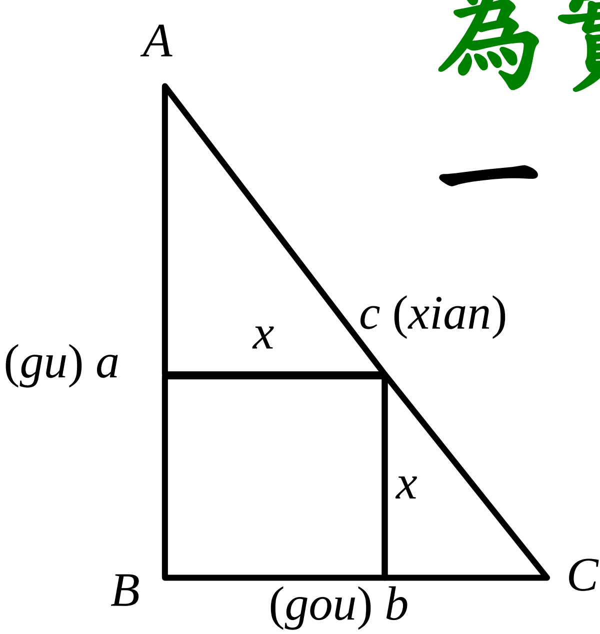
$$D'E' = DE$$

《九章算術》[成書於公元前一世紀至公元一世紀之間]
第九章第十五題


今有句五步股十二步問句中容方幾何答曰方三步
十七分步之九
術曰并句股為法句股相乘為實實如法而一得方 案方字下原本衍一步二字乃後人妄加今刪正
句股相乘為朱青黃冪各二 案此及下注舊有圖而佚今各補圖于後
令黃冪袤于隅中朱青各以其類令從其兩徑共成
脩之冪 案此有訛舛據從容圓術注云可用畿于小纸分裁邪正之會令顛倒相補各以類合成脩冪則此亦謂令黃冪連于下隅朱青各以其類移而相當共成脩冪也 方中黃 案此三字下有股文當云中方黃為廣
并句股為袤故并句股為法冪圖方在
九章算術 卷九
句中 案圖字誤當作圖 則方之兩廉各自成小股袤 案此句誤當云各自成小句股
而其相與之勢不失本率也句中之小股股
面之并為中率 案此亦訛舛當是言句面之小股股面之小句從橫相連合而成中方
令股為中方率并句股為率 案此四字之下有股文據當云并句股為廣率
見句五步而今有之得中方也復令句為中率以句
股為率 案此二句有脫誤當云復令句為中方率以并句股為袤率 據股十二步
而今有之則中方又可知 案此句舛誤當云則中方又可知 此則雖
不效而法實有法由生矣 案此亦舛誤據上以粟米章今有術及衰分章列衰之意解此術大小句股互求并句股即所有率中方率即所求率見句見股即所有數于事雖不同而意
相做效實術所由生也注意當是如此 不容圓率而似今有衰分言之 案此二句舛誤當云下容圓率而以今有衰分言之 可以見之也

「今有句五步，股十二步。問：句中容方幾何。答曰：方三步一十七分步之九。」

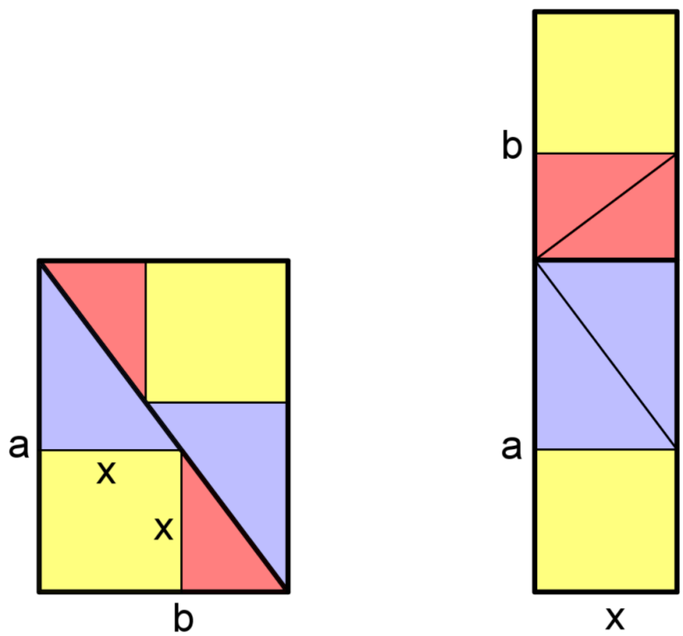
「**術曰**：并句、股為法，句股相乘為實，實如法而一，得方一步。」



$$\boxed{x = \frac{ab}{a+b}}$$

Commentary by LIU Hui (劉徽) [mid 3rd century]

Method **(dissect-and-re-assemble)**



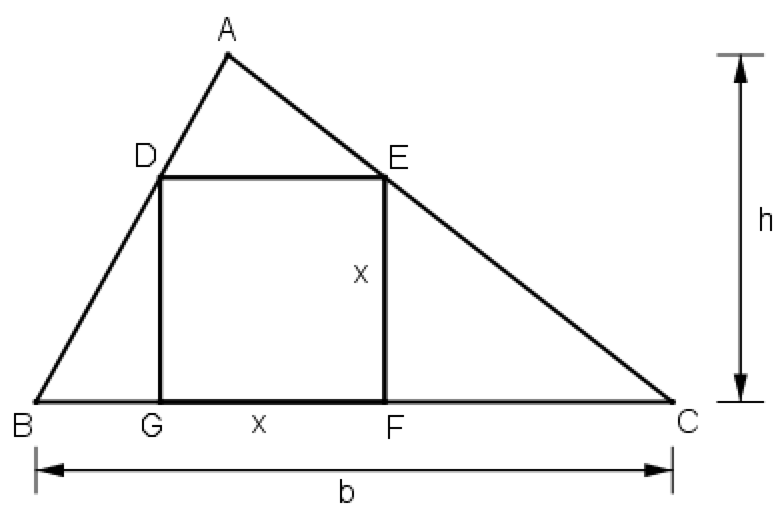
Area = ab Area = $(a + b)\,x$

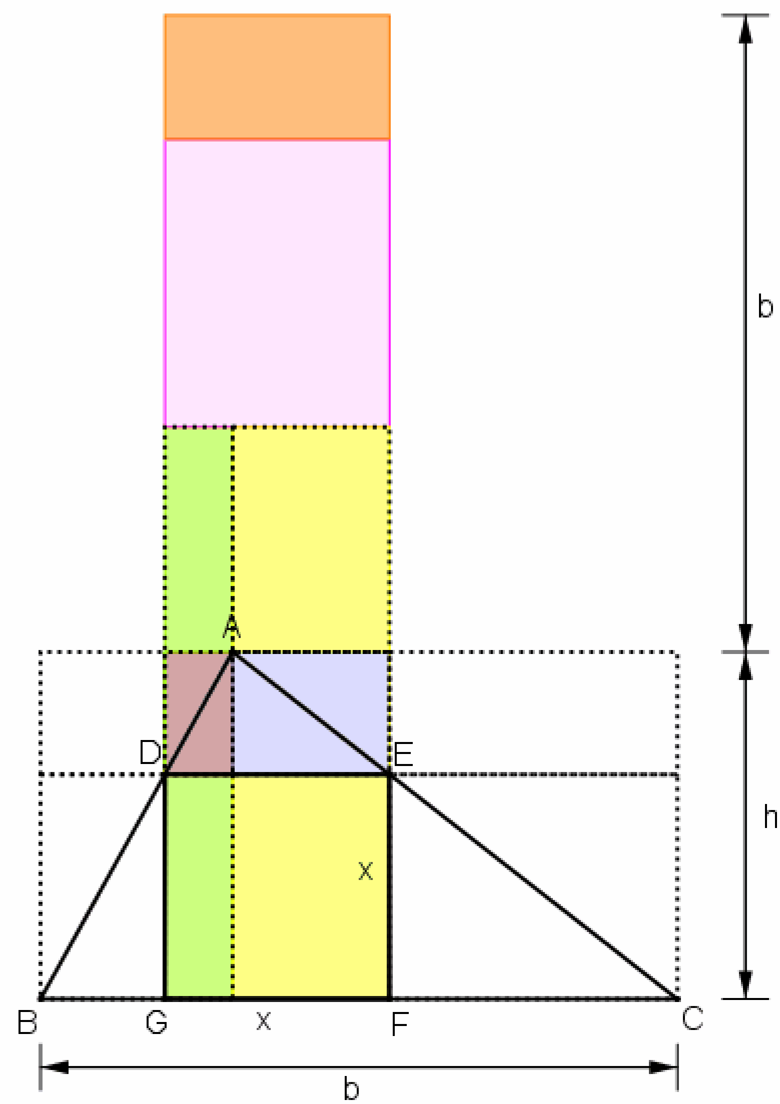
$$ab = (a + b)\,x$$

$$\boxed{x = \frac{ab}{a+b}}$$

http://ggbtu.be/m2812253

Calculate x in terms of b, h.

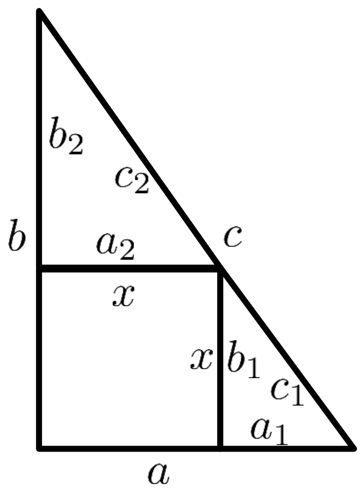




$$bh = x(b + h)$$

$$x = \frac{bh}{b + h}$$

Alternative proof of the formula in Problem 15 of Chapter 9 *of Jiuzhang Suanshu* (LIU Hui)



"To the top and to the right of the square there appear respective smaller right triangles. The relations between their sides retain the same rates as in the original triangle."

方在勾中，則方之兩廉各自成小勾股，而其相與之勢不失本率也。

$$a : b : c = a_1 : b_1 : c_1 = a_2 : b_2 : c_2.$$

$$\text{Hence, } \frac{a+b}{b} = \frac{a_1+b_1}{b_1} = \frac{a}{x},$$

$$\therefore \quad x = \frac{ab}{a+b}.$$

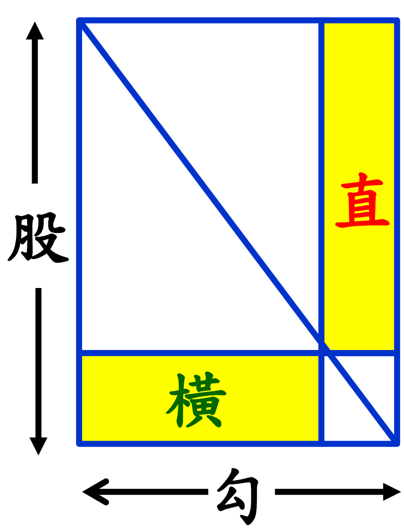
$$\left[\begin{array}{ll} \text{or} & \dfrac{a}{a+b} = \dfrac{a_2}{a_2+b_2} = \dfrac{x}{b}, \\[2ex] & \therefore \quad x = \dfrac{ab}{a+b}. \end{array} \right]$$

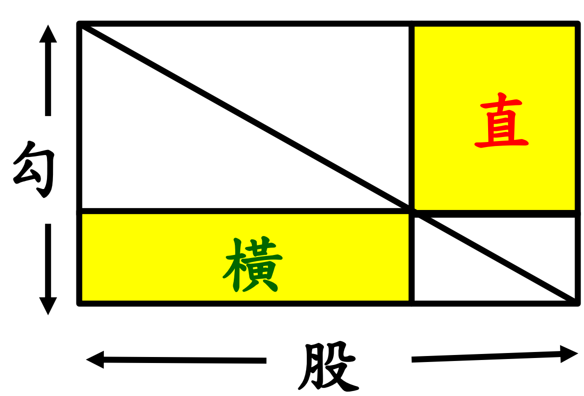
勾(股)中容橫。股(勾)中容直。
二積皆同。古人以題易名。
若非釋名。則無以知其源。

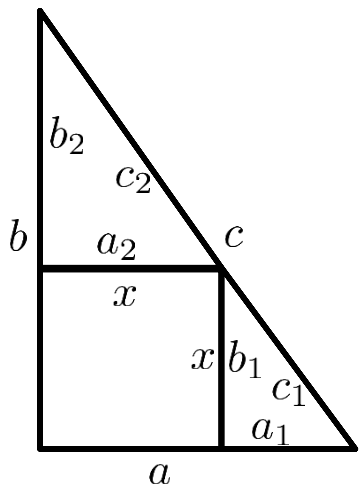
(The horizontal rectangle formed by part of the base and the vertical rectangle formed by part of the perpendicular are equal in area. Men of the past changed the names of their methods from problem to problem …)

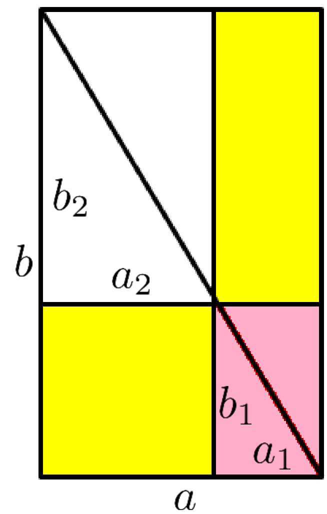
Compare with Proposition 43 Of Book I of Euclid's *Elements*.

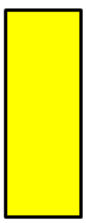
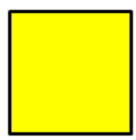
楊輝，《續古摘奇算法(卷下)》
YANG Hui, *Continuation of Ancient Mathematical Methods for Elucidating the Strange* [*Properties of Numbers*] (Chapter II) (1275)



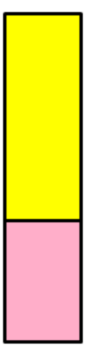
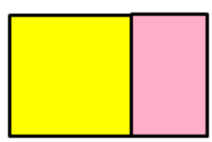






 $= a_1 b_2,$  $= a_2 b_1.$

Hence, $a_1 b_2 = a_2 b_1$, or $a_1 : a_2 = b_1 : b_2$.

 $= a_1 b,$  $= a b_1.$

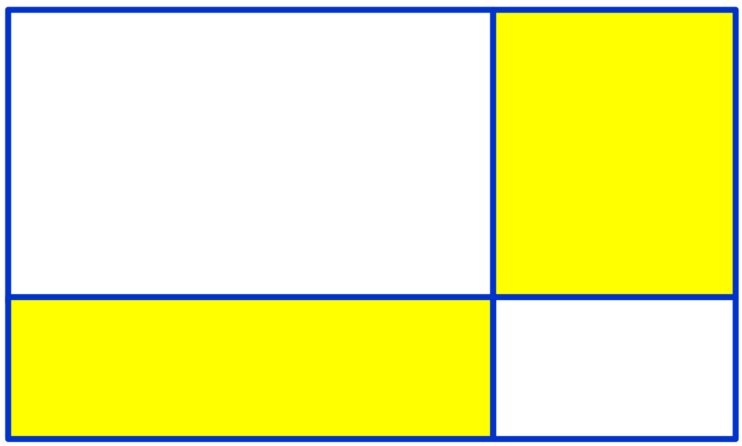
Hence, $a_1 b = a b_1$, or $a : a_1 = b : b_1$.

Since $c^2 = a^2 + b^2, c_1^2 = a_1^2 + b_1^2, c_2^2 = a_2^2 + b_2^2$,

we have $a : a_1 : a_2 = b : b_1 : b_2 = c : c_1 : c_2$,

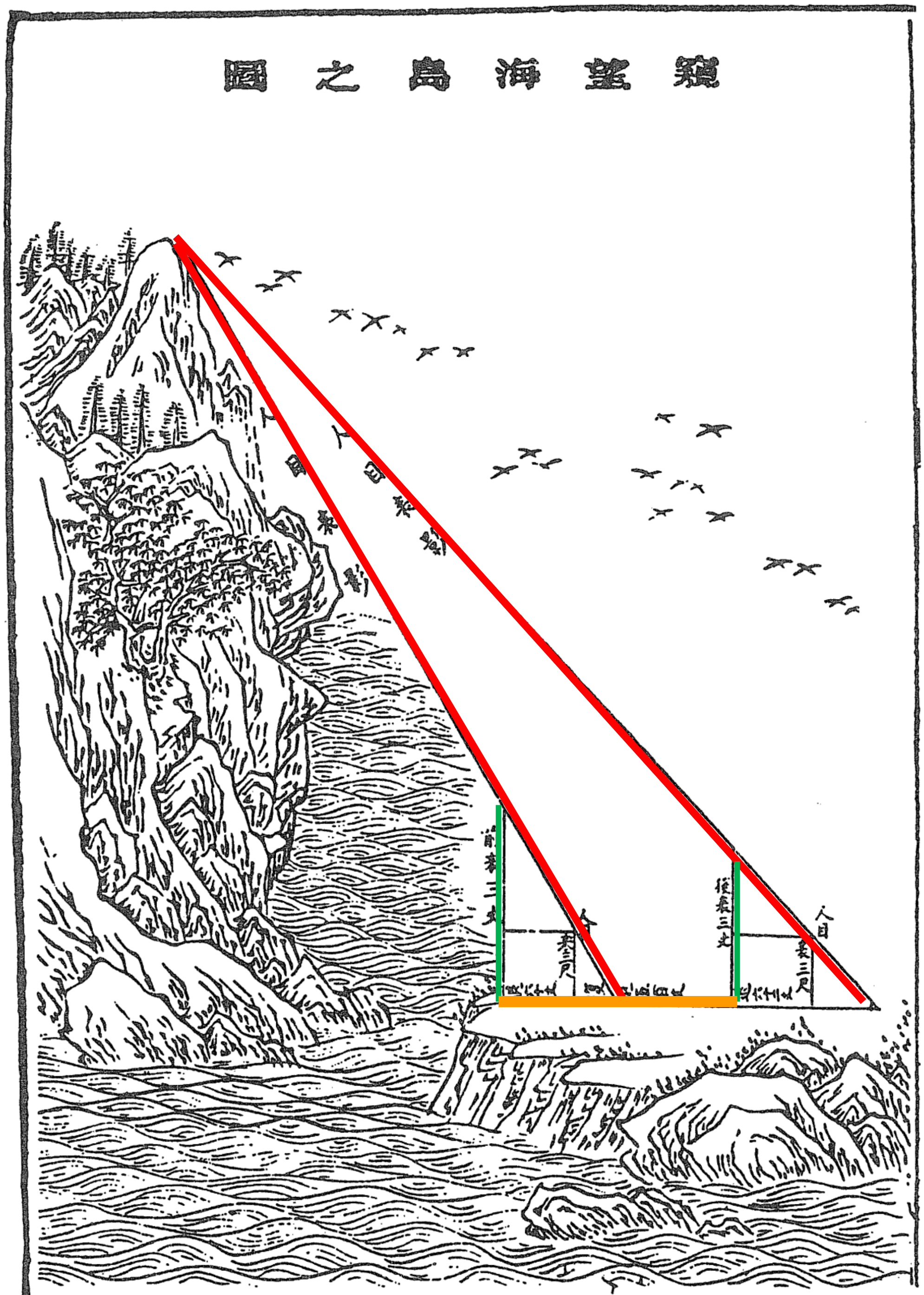
or $\qquad a : b : c = a_1 : b_1 : c_1 = a_2 : b_2 : c_2.$

A pedagogical extension to a locus problem (but with no historical context)



Question: When (and only when) will the two regions have equal area?

http://ggbtu.be/m2467811



LIU Hui's Method of Double-Difference in *Haidao Suanjing* [海島算經 Sea Island Mathematical Manual] (3rd century) as illustrated in *Gujin Tushu Jicheng* [古今圖書集成 Complete Collection of Pictures and Writings of Ancient and Modern Times] (1726)

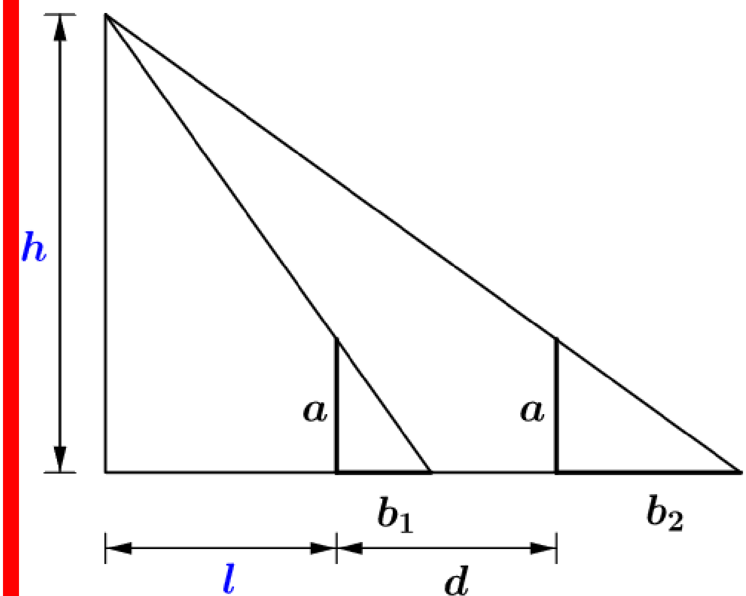
海島題解

魏劉徽註九章立重差著於勾股之下以闡世術夫度高測深勾股之法則無自而可知故重表累矩三望四望旁求審察是以崧山高下方邑大小其重表也岸望谷深山望津廣其累矩也登望崧高遙望波口非三望之術乎清淵自石登山臨邑非四望之術乎海島去表爲之篇首因以名之實九章勾股之遺法也迄今千餘載間唐李淳風而續算草未聞解白作法之旨者輝嘗置海島小圖於座右乃見先賢作法之萬一若欲盡傳豈不輕易祕旨或不傳流亦無以伸前賢之美本經題目廣遠難於引證學者非之今將孫子度影量竿題問引用詳解以驗小圖姑以一問其餘好學君子自能觸類而攷何必輕傳

凡股中容橫句中容直二積皆同古人以題易名若非釋名則無以知其源

楊輝在《續古摘奇算法》(1275)
對劉徽的重差術（《海島算經》）
作了解釋

Given a, d, b_1 and b_2, how can we express h and l in terms of a, d, b_1 and b_2 ?



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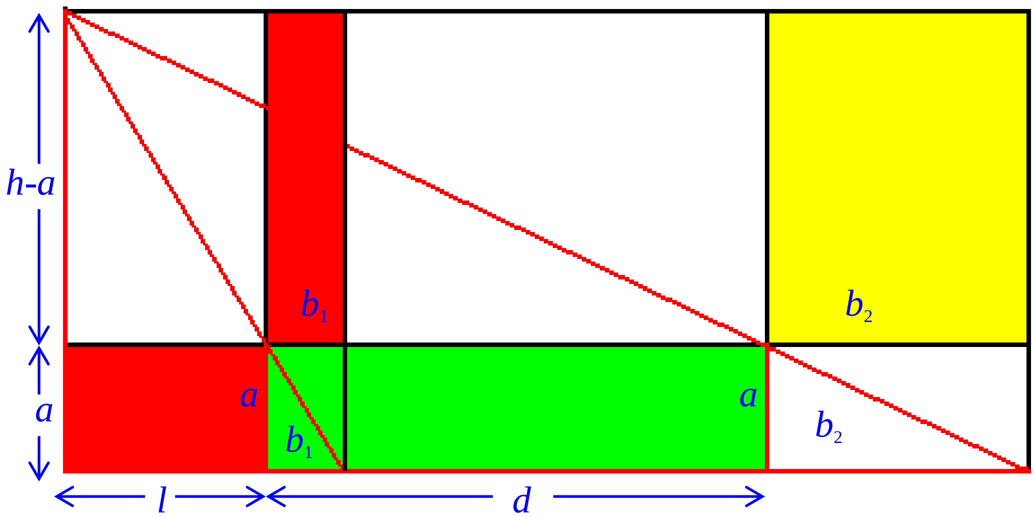
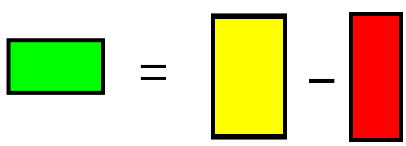
楊輝對劉徽的重差術的解釋

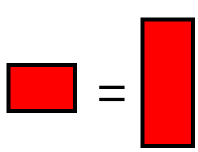
(1275)





$$
\begin{aligned}
ad &= b_2(h-a) - b_1(h-a) \\
&= (b_2 - b_1)(h-a)
\end{aligned}
$$

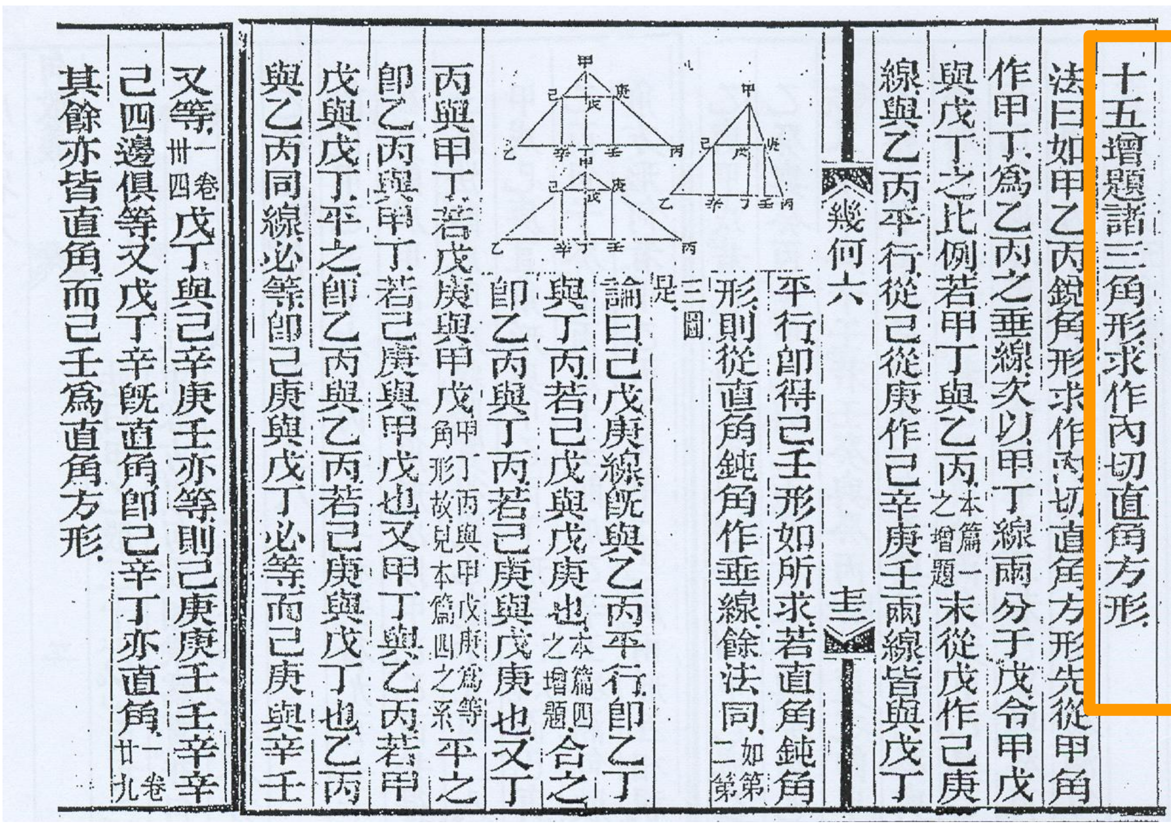
$$h = \frac{ad}{b_2 - b_1} + a$$

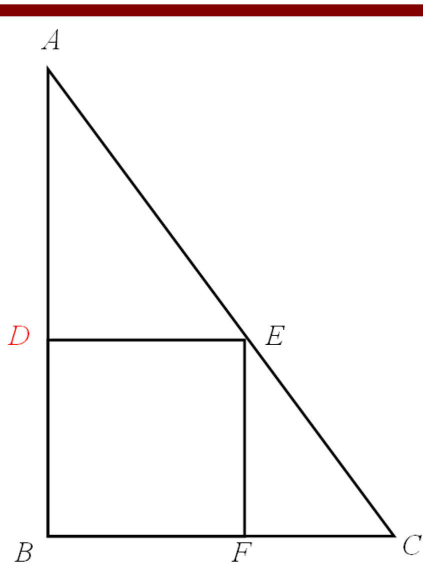


$$la = b_1(h-a) = \frac{b_1 ad}{b_2 - b_1}$$

$$l = \frac{b_1 d}{b_2 - b_1}$$

這題目的更一般型式，見諸利徐二氏翻繹的《幾何原本》(Clavius 本) 卷六的附加命題十五。


十五增題諸三角形求作內切直角方形
法曰如甲乙丙鈍角形求作內切直角方形先從甲角作甲丁為乙丙之垂線次以甲丁線兩分于戊令甲戊與戊丁之比例若甲丁與乙丙（本篇增題一）末從戊作己庚線與乙丙平行從己從庚作己辛庚壬兩線皆與戊丁平行即得己壬形如所求若直角鈍角形則從直角鈍角作垂線餘法同（如第二第三圖）
幾何六
論曰己戊庚線既與乙丙平行即乙丁與丁丙若己戊與戊庚也（本篇增題四）合之即乙丙與丁丙若己庚與戊庚也又丁丙與甲丁若戊庚與甲戊（甲丁丙與甲戊庚為等角形故見本篇四之系）平之即乙丙與甲丁若己庚與甲戊也又甲丁與乙丙若甲戊與戊丁平之即乙丙與乙丙若己庚與戊丁也乙丙與乙丙同線必等即己庚與戊丁必等而己庚與辛壬又等（卅四卷一）戊丁與己辛庚壬亦等則己庚庚壬壬辛辛己四邊俱等又戊丁辛既直角即己辛丁亦直角（廿九卷一）其餘亦皆直角而己壬為直角方形



以點 D 分割 AB，令 $AD : DB = AB : BC$ [卷六，命題 10]； 作 DE 平行 BC，EF 平行 AB，(E 在 AC 上，F 在 BC 上）；$DBFE$ 就是所需的內接正方形。

徐光啟，
《勾股義》
(1609)

句股義

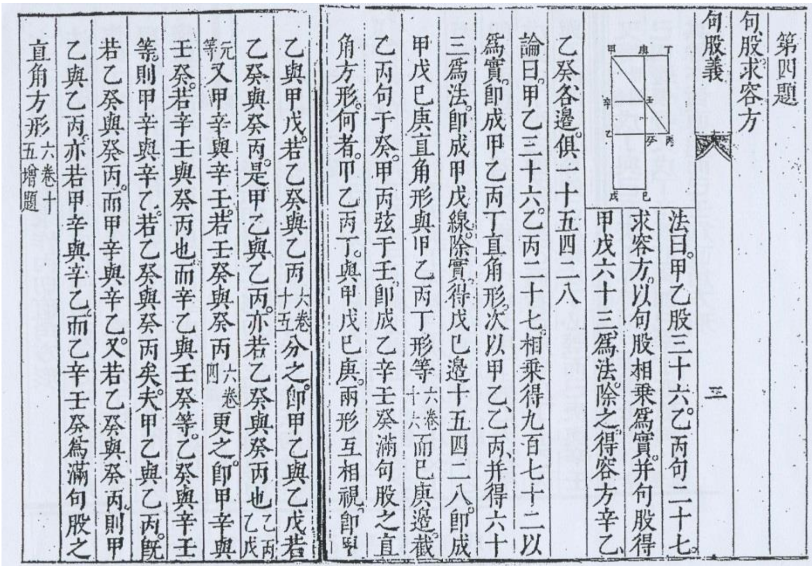
明 徐光啟 撰

句股即三邊直角形也底線為句底上之垂線為股對
直角邊為弦句股上兩直角方形并與弦上直角方形
等故句三股四則弦必五一卷四七注從此可以句股求弦
句弦求股股弦求句一卷四七注可以求句股中容方容圓
可以各較求句求

欽定四庫全書

句股自相求以至容方容圓各和各較相求者舊九章
中亦有之第能言其法不能言其義也所立諸法蕪陋
不堪讀門人孫初陽氏刪為正法十五條稍簡明矣余
因各為論譔其義使夫精於數學者覽圖誦說庶或為
之解頤

第能言其法
不能言其義也



第四題

句股求容方

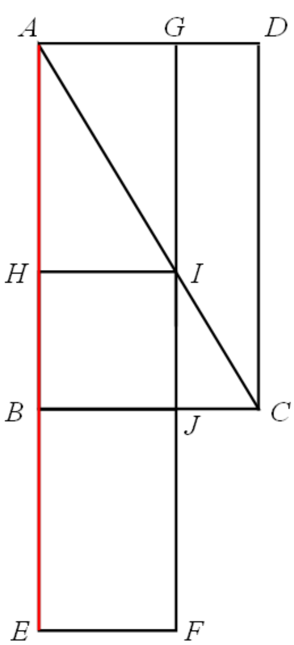
句股義　三

法曰甲乙股三十六乙丙句二十七
求容方以句股相乘為實并句股得
甲戊六十三為法除之得容方辛乙
乙癸各邊俱一十五四二八

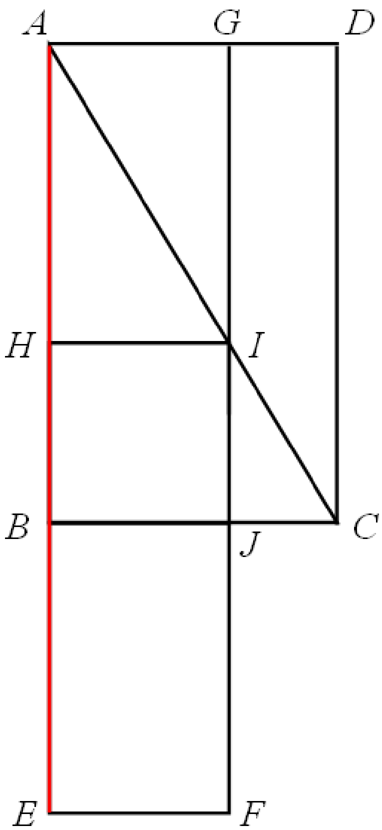
論曰甲乙三十六乙丙二十七相乘得九百七十二以
為實即成甲乙丙丁直角形次以甲乙乙丙并得六十
三為法即成甲戊線除實得戊己邊十五四二八即成
甲戊己庚直角形與甲乙丙丁形等 六卷十六 而己庚邊截
乙丙句于癸甲丙弦于壬即成乙辛壬癸滿句股之直
角方形何者甲乙丙丁與甲戊己庚兩形互相視即甲
乙與甲戊若乙癸與乙丙 六卷十五 分之即甲乙與乙戊若
乙癸與癸丙是甲乙與乙丙亦若乙癸與癸丙也 乙丙乙戊
元等 又甲辛與辛壬若壬癸與癸丙 六卷四 更之即甲辛與
壬癸若辛壬與癸丙也而辛乙與壬癸等乙癸與辛壬
等則甲辛與辛乙若乙癸與癸丙矣夫甲乙與乙丙既
若乙癸與癸丙而甲辛與辛乙又若乙癸與癸丙則甲
乙與乙丙亦若甲辛與辛乙而乙辛壬癸為滿句股之
直角方形 六卷十五增題

徐光啟，
《勾股義》(1609)

第四題： 句股求容方。法曰：甲乙股三十六，乙丙句二十七，求容方。以句股相乘為實，并句股得甲戊六十三為法，除之得容方辛乙、乙癸、各邊。俱一十五四二八。…



已知容方邊長為句乘股除以句加股。先製作 $AEFG$ 與 $ABCD$ 等面積，$AEFG$ 的一邊是句加股 $(BE = BC)$，另一邊便是求作容方的邊。由此證明點 H 分割 AB 滿足 $AH:HB = AB:BC$。符合附加命題十五的要求。



其實徐光啟是具備足夠綜合幾何知識去直接證明 $HB = HI$。

首先，$AEFG$ 與 $ABCD$ 等面積，故

$AB : EF = AE : BC$，

即 $AB : HI = AE : BC$。

然而，$AH : HI = AB : BC$，

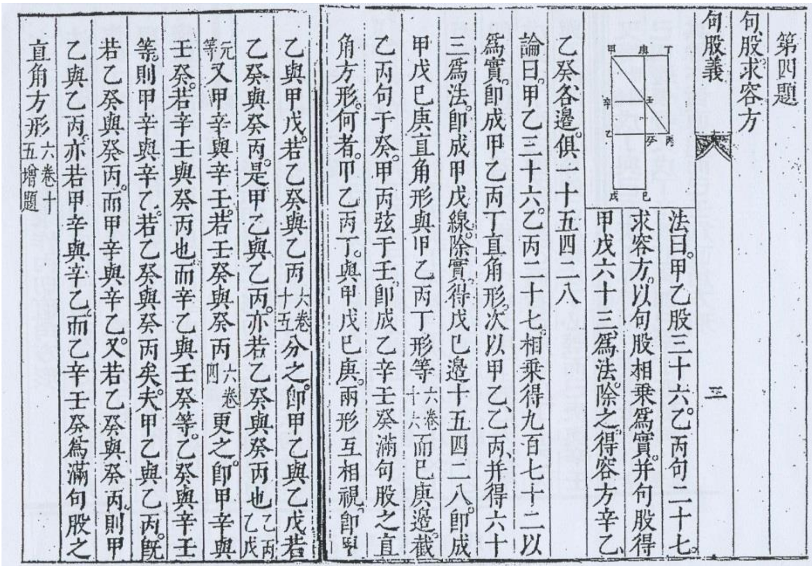
故 $AH+HI : HI = AB+BC : BC$，

即 $AH+HI : HI = AB+BE : BC$

$= AE : BC$，

因此 $AB = AH + HI$，

即 $HB = HI$。


第四題

句股求容方

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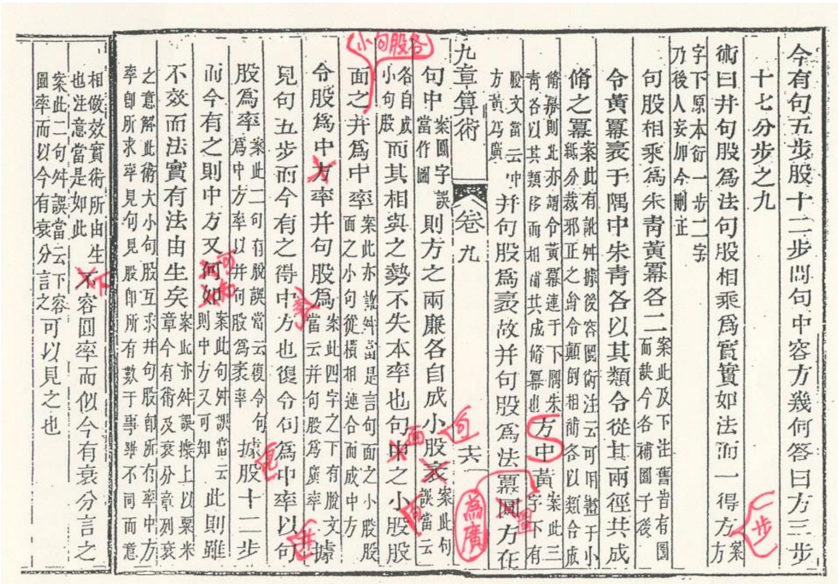
法曰甲乙股三十六乙丙句二十七求容方以句股相乘爲實并句股得甲戊六十三爲法除之得容方辛乙乙癸各邊俱一十五四二八

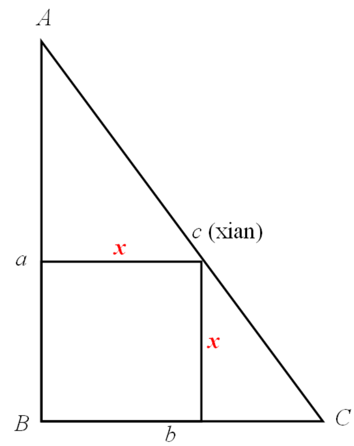
論曰甲乙三十六乙丙二十七相乘得九百七十二以爲實即成甲乙丙丁直角形次以甲乙乙丙并得六十三爲法即成甲戊線除實得戊己邊十五四二八即成甲戊己庚直角形與甲乙丙丁形等六卷十六而己庚邊截乙丙句于癸甲丙弦于壬即成乙辛壬癸滿句股之直角方形何者甲乙丙丁與甲戊己庚兩形互相視即甲乙與甲戊若乙癸與乙丙六卷十五分之即甲乙與乙戊若乙癸與癸丙是甲乙與乙丙亦若乙癸與癸丙也乙丙乙戊元等又甲辛與辛壬若壬癸與癸丙六卷四更之即甲辛與壬癸若辛壬與癸丙也而辛乙與壬癸等乙癸與辛壬等則甲辛與辛乙若乙癸與癸丙矣夫甲乙與乙丙既若乙癸與癸丙而甲辛與辛乙又若乙癸與癸丙則甲乙與乙丙亦若甲辛與辛乙而乙辛壬癸爲滿句股之直角方形六卷十五增題

徐光啟，
《勾股義》(1609)

徐光啟在書中用到的
複雜推論，看來
迂迴而且非必要。
可能，這顯示了西方
與中國處理數學的方式
有某種不協調，
勉強把一種方式塑造
成另一種，硬套進去，
便顯得很不自然了。

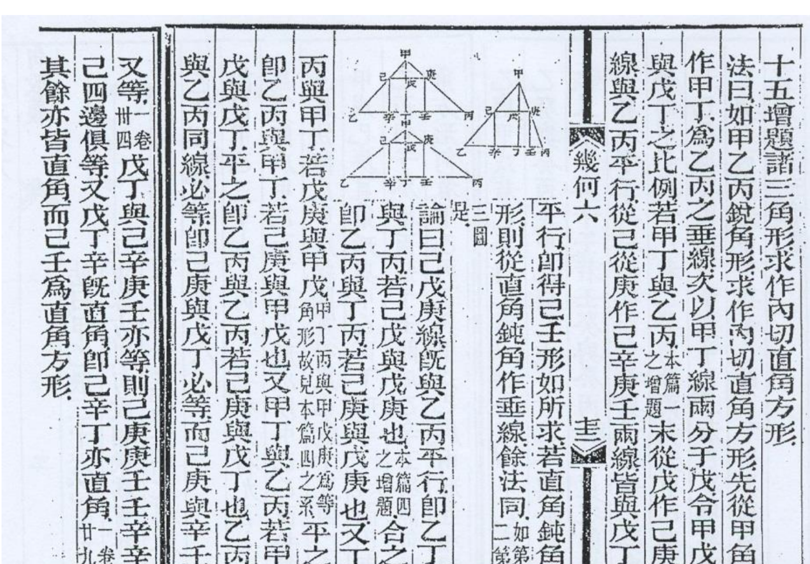
Problem 15 in Chapter 9 of *Jiuzhang Suanshu* (九章算術)

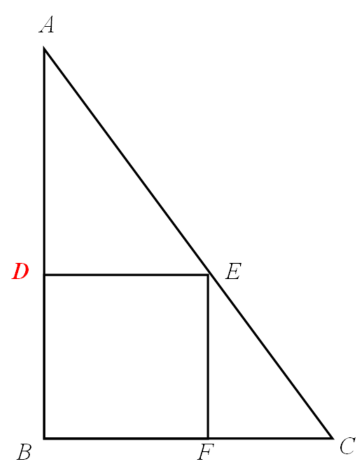




$$x = \frac{ab}{a+b}$$

Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV*





$AD : DB = AB : BC$

I was brought up [in school] with a large dose of synthetic geometry replete with lots of proofs and construction problems.

- ► accustomed to the notion of **proof and logic** .
- ► the joy of discovery and the joy of succeeding in understanding something which was **tangible** (you can at least draw some pictures even if you do not know why it has to be like that at first) but **not obvious** (you do not know why it is like that at first).
- ► Geometry is a subject in which one can exercise **logical discipline** and **free imagination** at the same time .
- ► geometric viewpoint (**flexibility in framework**).

M.K. Siu, Learning and teaching of analysis in the mid twentieth century: A semi-personal observation, *One Hundred Years of L'Enseignement Mathématique*, ed. D. Coray *et al*, 2003, 177-190.

The plan adopted throughout is to develop each group of geometrical facts by the following successive stages: -

(i) *Examples for oral discussion*.

These are illustrated extensively by diagrams in order to simplify black-board work.

This oral work gives the pupil a clear understanding of the relevant facts, familiarizes him with the arguments which will be used later in the formal proofs of theorems, and trains him in methods for solving riders. It includes, when appropriate, questions in which the data are numerical.

(ii) *An exercise of numerical examples*.

……

(iii) *Formal proofs of the corresponding theorems*.

……

(iv) *An exercise of riders*.

……

Preface of Durell's *A New Geometry for Schools* (1939)

CONTENTS

NOTE This Geometry is issued complete and in parts. Full details of styles will be found facing the title page. The following is the Table of Contents of the whole book.

SYMBOLS - - - - - - - - - page xiii
TABLES - - - - - - - - - ,, xiv
PHOTOGRAPH OF REGULAR SOLIDS - - - - - ,, xvi

STAGE A GEOMETRY

FUNDAMENTAL CONCEPTS - - - - - - - 1
Lines, Points, Solids, Surfaces, p. 1; Simple Solids, p. 2.

USE OF INSTRUMENTS - - - - - - - 5
Ruler, p. 5; Compass, p. 9; Set-squares, p. 15; Protractor, p. 30; Construction of Surfaces, p. 82.

ANGLES - - - - - - - - - - 16
Right Angles, p. 16; Vertical and Horizontal, p. 20; Compass Directions, p. 22; Notation, p. 24; Angles at a Point, p. 25; Compass Bearings, p. 32.

PROPERTIES OF PARALLELS - - - - - - 35
Angles made by Transversal, p. 35; Tests for Parallel Lines, p. 38.

ANGLES OF A TRIANGLE - - - - - - - 42
Definitions, p. 42; Exterior Angle, p. 43; Angle Sum, p. 46.

ANGLES OF A POLYGON - - - - - - - 49
Definitions, p. 49; Angle Sum and Exterior Angles, p. 50.

CONGRUENT TRIANGLES - - - - - - - 52
Necessary Data, p. 52; Tests for Congruence, p. 53; Applications to Ruler and Compass Constructions, p. 61.

SCALE DIAGRAMS - - - - - - - - 64
Plans, p. 64; Heights and Distances, p. 68.

SIMILAR TRIANGLES - - - - - - - - 70
Necessary Data, p. 70; Tests for Similarity, p. 71.

TRIGONOMETRICAL RATIOS - - - - - - 75
Tangent of Angle, p. 75; Sine and Cosine, p. 78.

The work in Stage A gives practice in the use of instruments, and deals with the fundamental facts associated with parallels, congruence, and similarity.

Preface to Durell's *A New Geometry for Schools* (1939)

Construction of Surfaces of Solids

The *open* box which holds the matches in a Brymay match-box is 2·3 in. long, 1·5 in. wide, and 0·7 in. high. If you cut down the edges and fold the sides of the box down level with the base, you will obtain a figure of the shape represented in fig. 196, and this is called the net of the open box.

What are the lengths of the different parts of the boundary of fig. 196, if the dimensions of the open match-box are those given above?

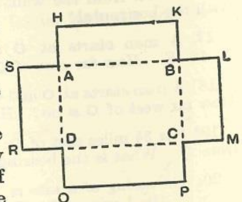


FIG. 196

Which lines in the net *must* be equal?

Draw out the net on stiff paper or thin cardboard; make creases along the dotted lines and then fold so as to obtain an open box. Use gummed paper to fasten the edges together.

EXERCISE 23

1. Make a sketch showing the net of a closed box, 5 cm. long, 3 cm. wide, 2 cm. high. Show the dimensions of the net on your sketch. Draw the figure accurately and construct the box.

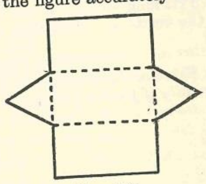


FIG. 197

2. Fig. 197 represents the net of a triangular prism. Show on your own figure the dimensions of the net if the prism is 5 cm. high and if each edge of the base is 3 cm. Construct the prism.

3. Fig. 198 represents the net of a regular tetrahedron, see the photograph opposite p. 1. Show on your own figure the dimensions of the net if each edge of the solid is 2 in. long. Construct the solid.

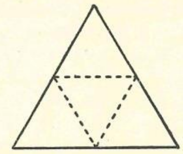


FIG. 198

4. Fig. 199 represents the net of a pyramid on a square base, see fig. 3, p. 2. Show on your own figure the dimensions of the net if each edge of the base is 3 cm. and each of the slant edges is 4 cm. Construct the pyramid.

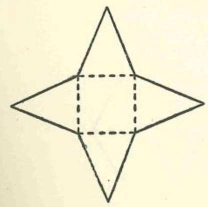


FIG. 199

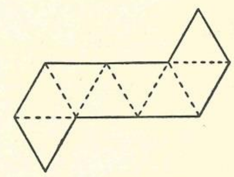


FIG. 200

5. Fig. 200 represents the net of a regular octahedron, each face of which is an equilateral triangle, see the photograph opposite p. 1. Show on your own figure the dimensions of the net if each edge of the solid is 4 cm. long. Construct the solid.

6. Draw on stiff paper or thin cardboard the net of a prism, 4 cm. long, whose base is a regular pentagon, see fig. 3, p. 2. To draw the pentagon, prick through the points marked 1, 2, 3, 4, 5 in fig. 11, p. 8. Construct the solid.

7. Fig. 201 represents the net of a regular dodecahedron, see the photograph opposite p. 1. Each part of the net is a regular pentagon. To draw a central pentagon, prick through the points marked 1, 2, 3, 4, 5 in fig. 11, p. 8. Then fit equal pentagons round the edges of this pentagon. Construct the solid.

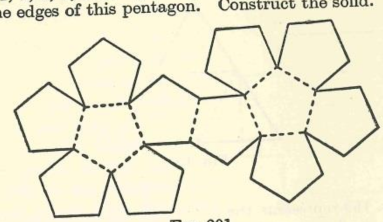


FIG. 201

8. Fig. 202 represents the net of a regular icosahedron, see the photograph opposite p. 1. Each part of the net is an equilateral triangle. Construct the solid.

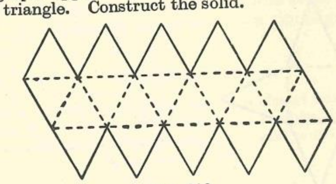


FIG. 202

9. A circular cylinder, see fig. 3, p. 2, is made of thin paper and has both ends closed. Its height is 6 cm. and its girth (*i.e.* the circumference of the base) is 11 cm. Draw the net from which it could be constructed. Use the fact that the circumference of a circle is $\frac{22}{7}$ times the diameter, approximately, to find the diameter of each circular end of the cylinder. Construct the cylinder.

10. Draw a semicircle of radius 6 cm. on stiff paper, *not* cardboard, and cut it out. Coil it so as to obtain the curved surface of a circular cone, see fig. 3, p. 2. Cut out also a circle of radius 3 cm.; this will form the base of the circular cone. Construct the solid.

C.V. Durell, *A New Geometry For Schools* : *Stage A* (1939).

Those teachers and examiners who are in a position to compare the results obtained by the teaching of Geometry in schools today with those obtained before the dethronement of Euclid agree almost unanimously that there have been both gain and loss. On the one hand, almost all pupils today acquire much more power in applying and reasoning from the fundamental facts of Geometry than did their predecessors, but, on the other hand, their reasoning is often less rigorous, and the average pupil often fails lamentably to reproduce the standard proofs when called upon in examinations.

Almost all would agree that the gain outweighs the loss; for the educational value of the subject lies far more in the former than in the latter accomplishment. … the loss need not accompany the gain, ….

THE ESSENTIALS

OF

SCHOOL GEOMETRY

BY

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First edition 1933

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON
1950

A.B. Mayne, Preface to *The Essentials of School Geometry* (1933)

PART I

ANGLES, CONGRUENCES, PARALLELS, IN-EQUALITIES, AND PARALLELOGRAMS

INTRODUCTORY

Geometry is the science of space, and consists of the study of the shapes, sizes and positions of objects which occupy space.

All reasoning is founded on certain simple principles, the truth of which is admitted without proof. These self-evident truths are called **Axioms**. The following is a list of axioms which apply to magnitudes of all kinds. Certain other axioms relating to geometrical magnitudes only will be stated later as they are required.

Axioms:

1. Things which are equal to the same thing are equal to one another.

2. If equals are added to equals, the sums are equal.

3. If equals are taken from equals, the remainders are equal.

4. If equals are added to unequals, the sums are unequal.

5. If equals are taken from unequals, the remainders are unequal.

6. Things which are the same multiples of equals are equal to one another, e.g. Doubles of equals are equal to one another.

7. Things which are the same parts of equals are equal to one another, e.g. Quarters of equals are equal to one another.

8. The whole is greater than its part.

DEFINITIONS

Surface. The space occupied by any object (say a penny or a football) is limited by boundaries which separate it from surrounding space. These boundaries are called **surfaces.**

A **surface** has length and breadth, but no thickness.

Surfaces intersect (or meet) in **lines.**

A **line** has length, but no breadth. Lines intersect in **points.**

A **point** has position, but no magnitude.

In practice a line is represented by the mark traced by a thin pencil-point on a sheet of paper, and a point is represented either by a small dot on the paper or, preferably, by the intersection of two small lines, e.g. X.

Straight lines. A line may be straight or curved. Everyone knows what is meant by a straight line, and it is almost impossible to add to this knowledge by a formal definition. The following is the definition usually given:

A straight line is that which lies evenly between its extreme points.

Axiom. Two straight lines cannot enclose a space.

It follows that two straight lines cannot intersect in more than one point. For, if they met in two points, they would enclose a space.

POSTULATES

In order to draw the figures required for the study of geometry certain instruments are required. These are (i) a straight ruler, (ii) a pair of compasses. It is also necessary to assume the possibility of performing certain simple operations by using these instruments.

A geometrical construction which we can take for granted that we can perform is called a **postulate.** The following postulates concern a straight line:

Postulate 1. Let it be granted that a straight line may be drawn from one point to any other point.

THE COMPARISON OF TWO TRIANGLES

Two figures which can be made to coincide are called **congruent.** Congruent figures are said to be **equal in all respects.** In two congruent figures, the sides and angles which coincide, when one figure is applied to the other, are said to **correspond.** They are also called **corresponding sides** and **corresponding angles** respectively.

In two congruent triangles corresponding sides are opposite equal angles, and corresponding angles are opposite equal sides.

It should be noted that, in order to make two triangles coincide, it may be necessary to turn one of them over.

* THEOREM 1

If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.

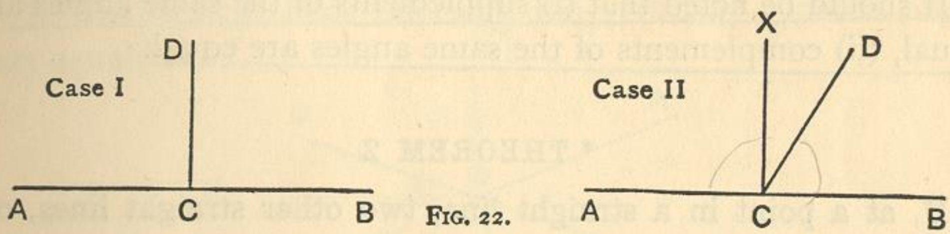


FIG. 22.

Let the straight line CD stand on the straight line AB. It is required to prove that the angles ACD, DCB are together equal to two right angles.

Case I. If the angles ACD, DCB are equal, each is by definition a right angle and they are together equal to two right angles.

Case II. If the angles ACD, DCB are unequal, let CX represent a line drawn through C perpendicular to AB.

Proof. $\angle ACD + \angle DCB$

$= \angle ACX + \angle XCD + \angle DCB$

$= \angle ACX + \angle XCB$

$= 2$ right angles, by construction.

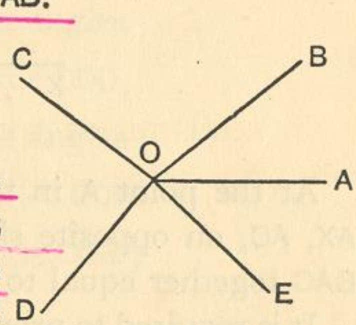


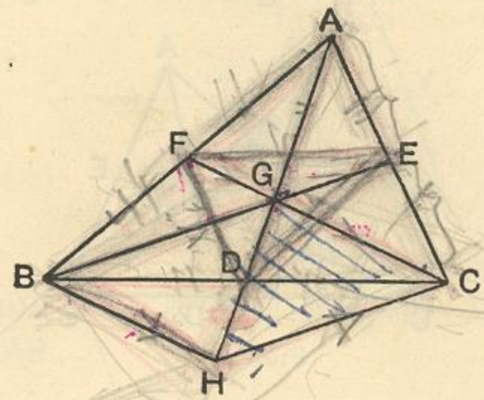
FIG. 23.

Corollary. If any number of straight lines are drawn from a given point, the sum of the consecutive angles so formed is equal to four right angles, i.e. in the figure

$\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA = 4$ right angles.

THEOREM C

The three medians of a triangle are concurrent.



diagonal bisect each other

Fig. 117.

Let ABC be a △, and let E, F be the mid-points of AC, AB. Let the straight lines BE, CF meet at G.

Join AG and produce it to cut BC at D.

It is required to prove that BD = DC.

Construction. Produce AG to H, making GH = AG. Join BH, CH.

Proof. In the △ ABH, FG is, by construction, the line joining the mid-points of the sides AB, AH;

∴ it is || to the base BH; ∴ FGC is || to BH.

Similarly, in the △ AHC, the line EGB may be proved to be || to CH,

∴ Both pairs of opposite sides of the figure BGCH are parallel;

∴ BGCH is a parallelogram.

But the diagonals of a parallelogram bisect one another;

∴ BD = DC.

Definition. The point of intersection of the medians is called the **centroid** of the triangle.

Corollary. The centroid is a third of the way up the median DA, measured from D.

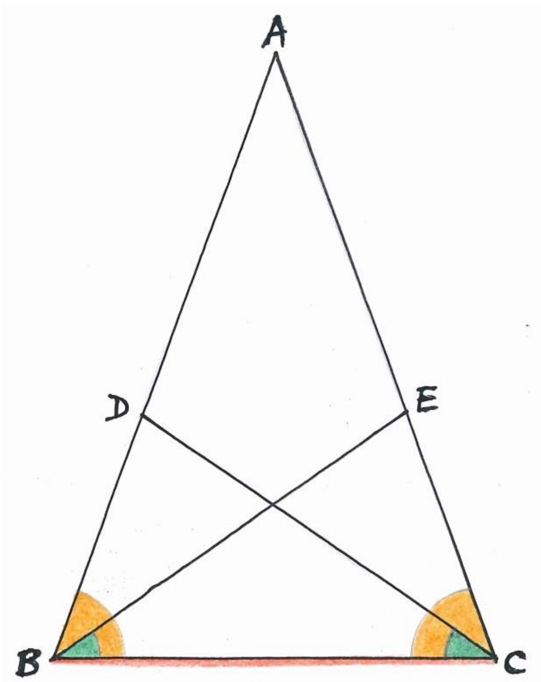
For, in the above figure, AG = GH, by construction.

But, since the diagonals of a parallelogram bisect one another;

$\therefore$ GD = DH, i.e. GD = $\frac{1}{2}$GH,

$\therefore$ GD = $\frac{1}{2}$AG, i.e. GD = $\frac{1}{3}$AD.

Similarly, it may be proved that GE = $\frac{1}{3}$BE, GF = $\frac{1}{3}$CF.



若 *ABC* 是等腰三角形，則底角的角平分線相等。

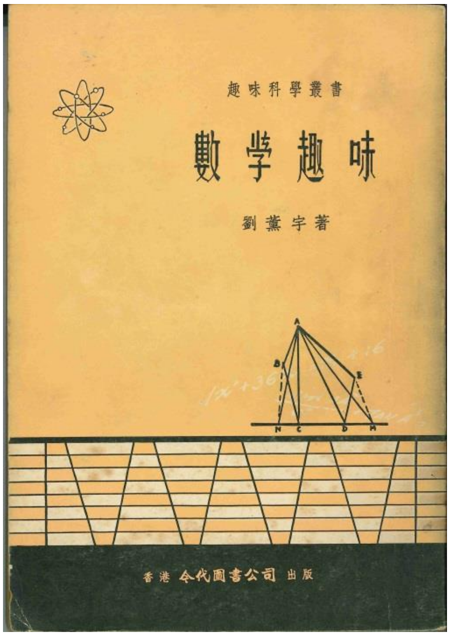
其逆定理是否成立?

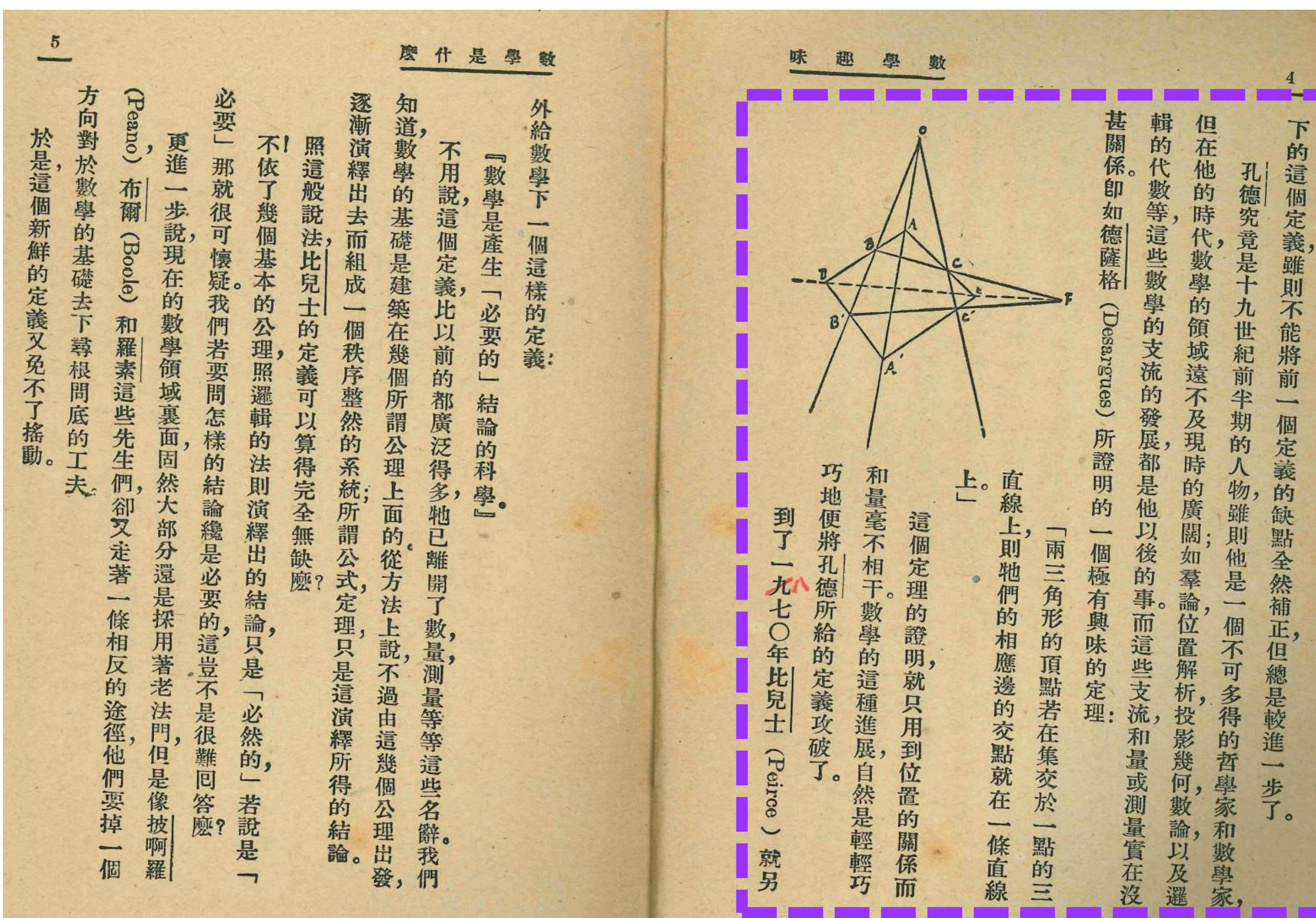
Steiner-Lehmus 定理

若三角形 *ABC* 底角的角平分線相等，則 *ABC* 是等腰。

在1840年 C.L. Lehmus 向 Jacob Steiner 提出這個問題， Steiner 解答了，但沒有即時發表。Lehmus 自己也於1850年解答了，並發表了證明，但少為人知 。在 1963年 G. Gilbert 和 D. MacDonnell 在 *American Mathematical Monthly* 上發表了一個很簡潔的證明，原來即是一百多年前 Lehmus 的證明！

劉薰宇,《*數學趣味*》,
香港今代圖書公司, 1958
(原版, 開明書店, 1933)





下的這個定義，雖則不能將前一個定義的缺點全然補正，但總是較進一步了。

孔德究竟是十九世紀前半期的人物，雖則他是一個不可多得的哲學家和數學家，但在他的時代，數學的領域遠不及現時的廣闊；如羣論，位置解析，投影幾何，數論，以及邏輯的代數等，這些數學的支流的發展，都是他以後的事。而這些支流，和量或測量實在沒甚關係。卽如德薩格（Desargues）所證明的一個極有興味的定理：

「兩三角形的頂點若在集交於一點的三直線上，則牠們的相應邊的交點就在一條直線上。」

這個定理的證明，就只用到位置的關係而和量毫不相干。數學的這種進展，自然是輕輕巧巧地便將孔德所給的定義攻破了。

到了一九七〇年比兒士（Peirce）就另

外給數學下一個這樣的定義：

『數學是產生「必要的」結論的科學。』

不用說，這個定義，比以前的都廣泛得多，牠已離開了數，量，測量等等這些名辭。我們知道，數學的基礎是建築在幾個所謂公理上面的。從方法上說，不過由這幾個公理出發，逐漸演繹出去而組成一個秩序整然的系統；所謂公式，定理，只是這演繹所得的結論。

照這般說法，比兒士的定義可以算得完全無缺麼？不！

不依了幾個基本的公理，照邏輯的法則演繹出的結論，只是「必然的，」若說是「必要」那就很可懷疑。我們若要問怎樣的結論纔是必要的，這豈不是很難回答麼？

更進一步說，現在的數學領域裏面，固然大部分還是採用著老法門，但是像披啊羅（Peano），布爾（Boole）和羅素這些先生們，卻又走著一條相反的途徑，他們要掉一個方向對於數學的基礎去下尋根問底的工夫。

於是，這個新鮮的定義又免不了搖動。

起首章：數學是什麼？

Desargues' Theorem

TRIANGLES IN PERSPECTIVE.

Definition. Two figures are said to be in perspective if the joins of corresponding pairs of points are all concurrent.

THEOREM 61.

(DESARGUES' THEOREM*.)

If two triangles are such that the lines joining their vertices in pairs are concurrent, then the intersections of corresponding sides are collinear.

若ΔABC 及 ΔA'B'C' 有透視中心 **(in perspective centrally)**，則必有透視軸 **(in perspective axially)**；反之亦然。

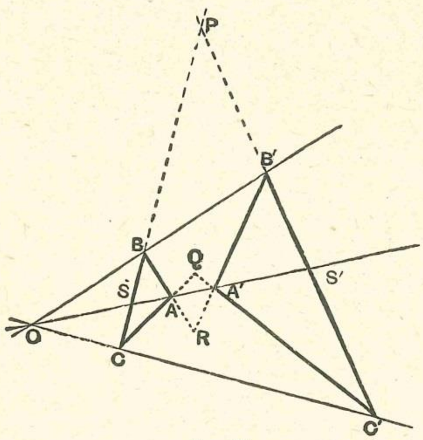


fig. 89.

The triangles ABC, A'B'C' are such that AA', BB', CC' meet at O.

Let BC, B'C' meet at P; CA, C'A' at Q; AB, A'B' at R. Let OAA' cut BC in S, B'C' in S'.

* Gerard Desargues (born at Lyons, 1593; died, 1662).

To prove that PQR is a straight line.

{PBSC} = {PB'S'C'} as both ranges lie on the pencil O {PBSC}.

$\therefore$ A {PBSC} = A' {PB'S'C'},

i.e. A {PROQ} = A' {PROQ}.

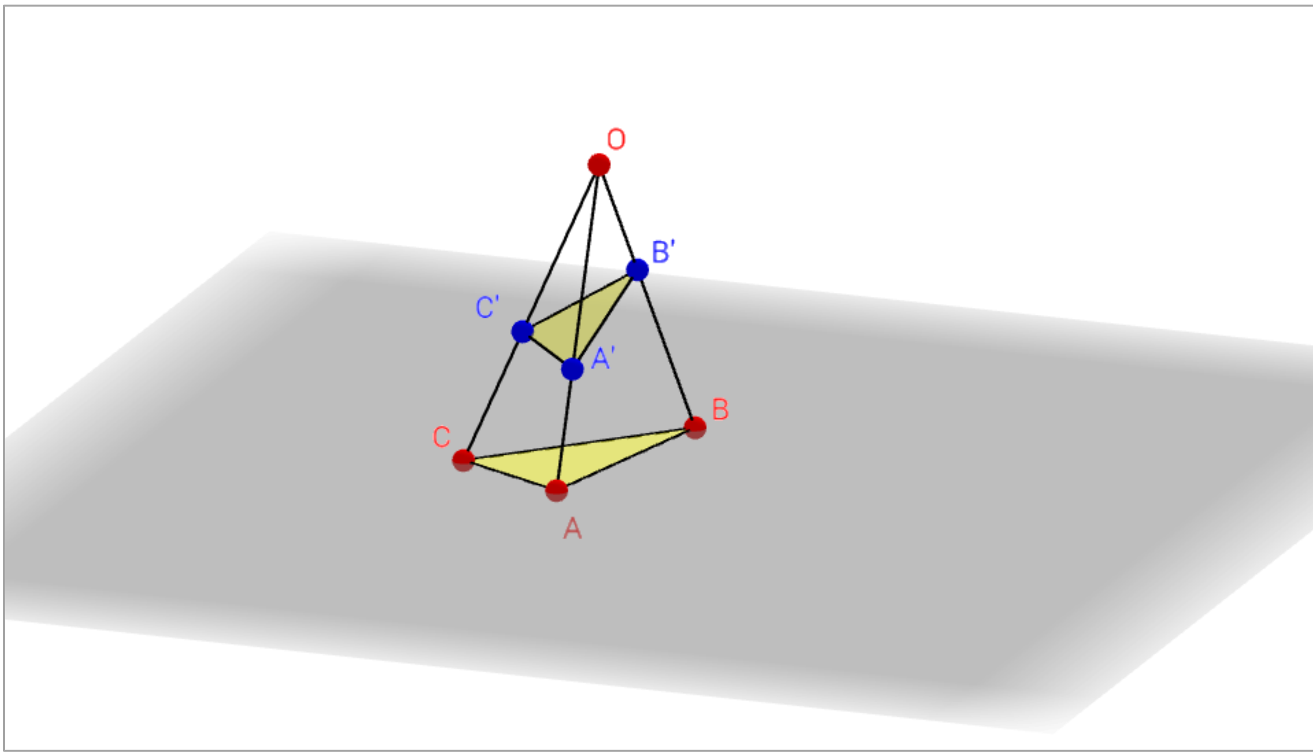
These two equicross pencils, therefore, have a line OAA' in common.

$\therefore$ P, Q, R are collinear. Th. 56.

Definition. The point O is called the centre of perspective, and the line PQR the axis of perspective of the two triangles ABC, A'B'C' in fig. 89.

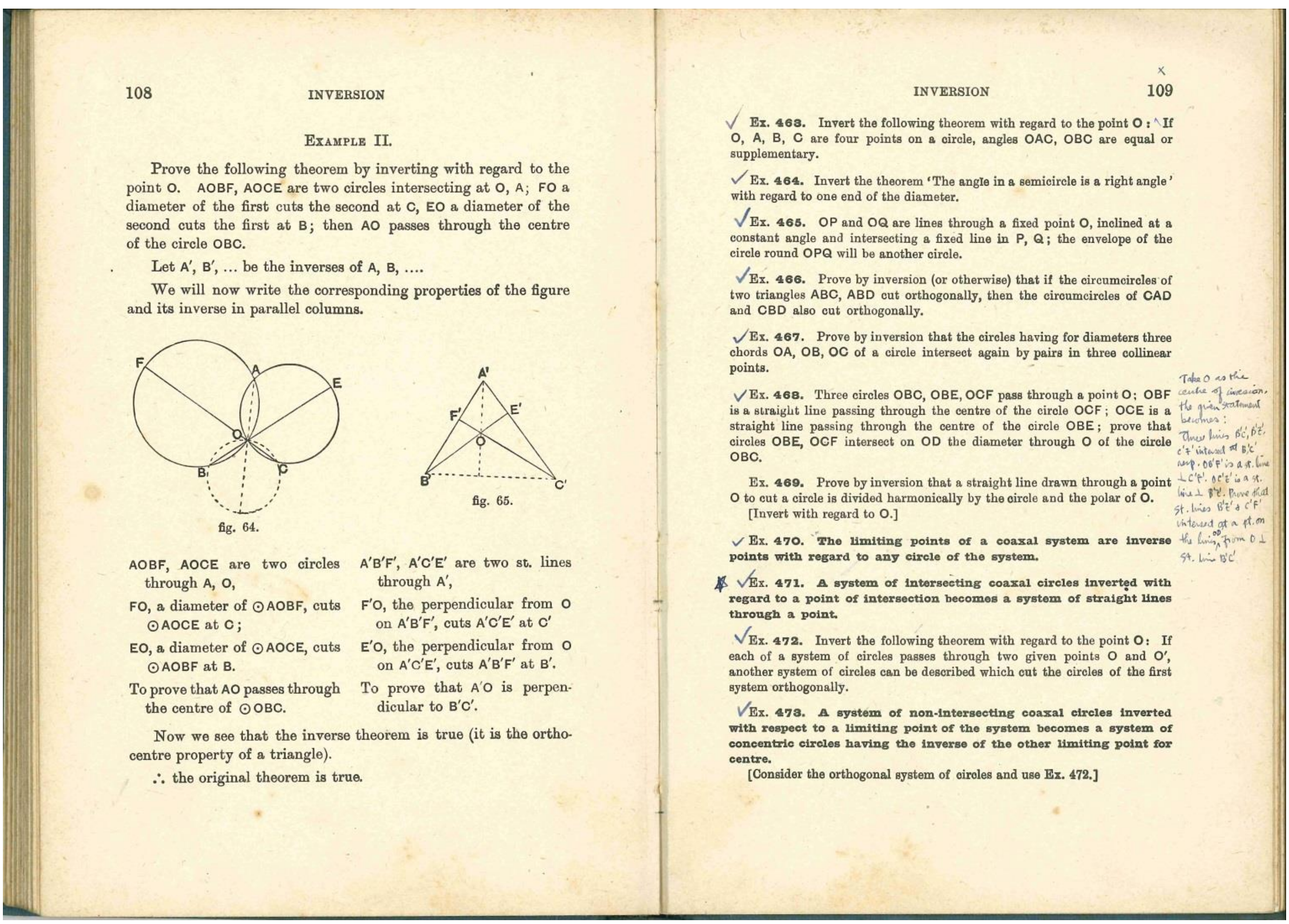
C. Godfrey, A.W. Siddons, *Modern Geometry*, Cambridge University Press, 1954; first published in 1908.

Desargues' Theorem (3D)



http://ggbm.at/6039247

Chapter XI. Inversion



EXAMPLE II.

Prove the following theorem by inverting with regard to the point O. AOBF, AOCE are two circles intersecting at O, A; FO a diameter of the first cuts the second at C, EO a diameter of the second cuts the first at B; then AO passes through the centre of the circle OBC.

Let A′, B′, … be the inverses of A, B, ….

We will now write the corresponding properties of the figure and its inverse in parallel columns.

fig. 64.

fig. 65.

AOBF, AOCE are two circles through A, O,	A′B′F′, A′C′E′ are two st. lines through A′,
FO, a diameter of ⊙AOBF, cuts ⊙AOCE at C;	F′O, the perpendicular from O on A′B′F′, cuts A′C′E′ at C′
EO, a diameter of ⊙AOCE, cuts ⊙AOBF at B.	E′O, the perpendicular from O on A′C′E′, cuts A′B′F′ at B′.
To prove that AO passes through the centre of ⊙OBC.	To prove that A′O is perpendicular to B′C′.

Now we see that the inverse theorem is true (it is the orthocentre property of a triangle).

∴ the original theorem is true.

Ex. 463. Invert the following theorem with regard to the point O: If O, A, B, C are four points on a circle, angles OAC, OBC are equal or supplementary.

Ex. 464. Invert the theorem 'The angle in a semicircle is a right angle' with regard to one end of the diameter.

Ex. 465. OP and OQ are lines through a fixed point O, inclined at a constant angle and intersecting a fixed line in P, Q; the envelope of the circle round OPQ will be another circle.

Ex. 466. Prove by inversion (or otherwise) that if the circumcircles of two triangles ABC, ABD cut orthogonally, then the circumcircles of CAD and CBD also cut orthogonally.

Ex. 467. Prove by inversion that the circles having for diameters three chords OA, OB, OC of a circle intersect again by pairs in three collinear points.

Ex. 468. Three circles OBC, OBE, OCF pass through a point O; OBF is a straight line passing through the centre of the circle OCF; OCE is a straight line passing through the centre of the circle OBE; prove that circles OBE, OCF intersect on OD the diameter through O of the circle OBC.

Take O as the centre of inversion, the given statement becomes: Three lines B′C′, B′E′, C′F′ intersect at B′, C′ resp. OB′F′ is a st. line ⊥ C′F′. OC′E′ is a st. line ⊥ B′E′. Prove that st. lines B′E′ & C′F′ intersect at a pt. on the line OD from O ⊥ st. line B′C′

Ex. 469. Prove by inversion that a straight line drawn through a point O to cut a circle is divided harmonically by the circle and the polar of O.

[Invert with regard to O.]

Ex. 470. **The limiting points of a coaxal system are inverse points with regard to any circle of the system.**

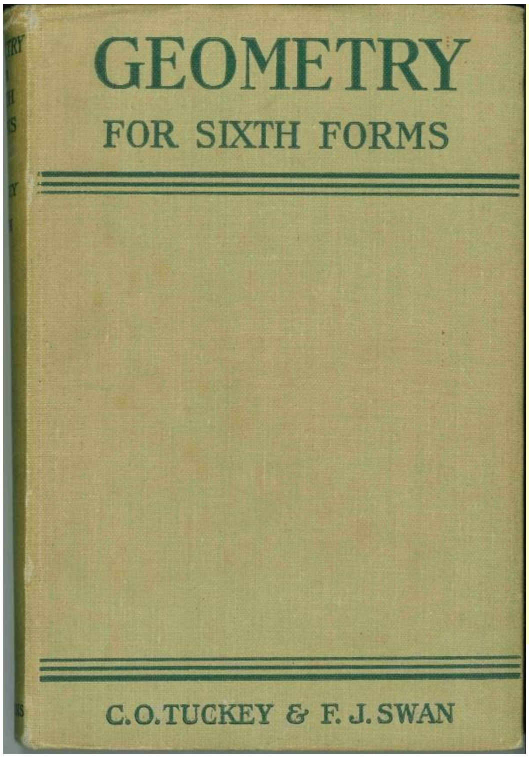
Ex. 471. **A system of intersecting coaxal circles inverted with regard to a point of intersection becomes a system of straight lines through a point.**

Ex. 472. Invert the following theorem with regard to the point O: If each of a system of circles passes through two given points O and O′, another system of circles can be described which cut the circles of the first system orthogonally.

Ex. 473. **A system of non-intersecting coaxal circles inverted with respect to a limiting point of the system becomes a system of concentric circles having the inverse of the other limiting point for centre.**

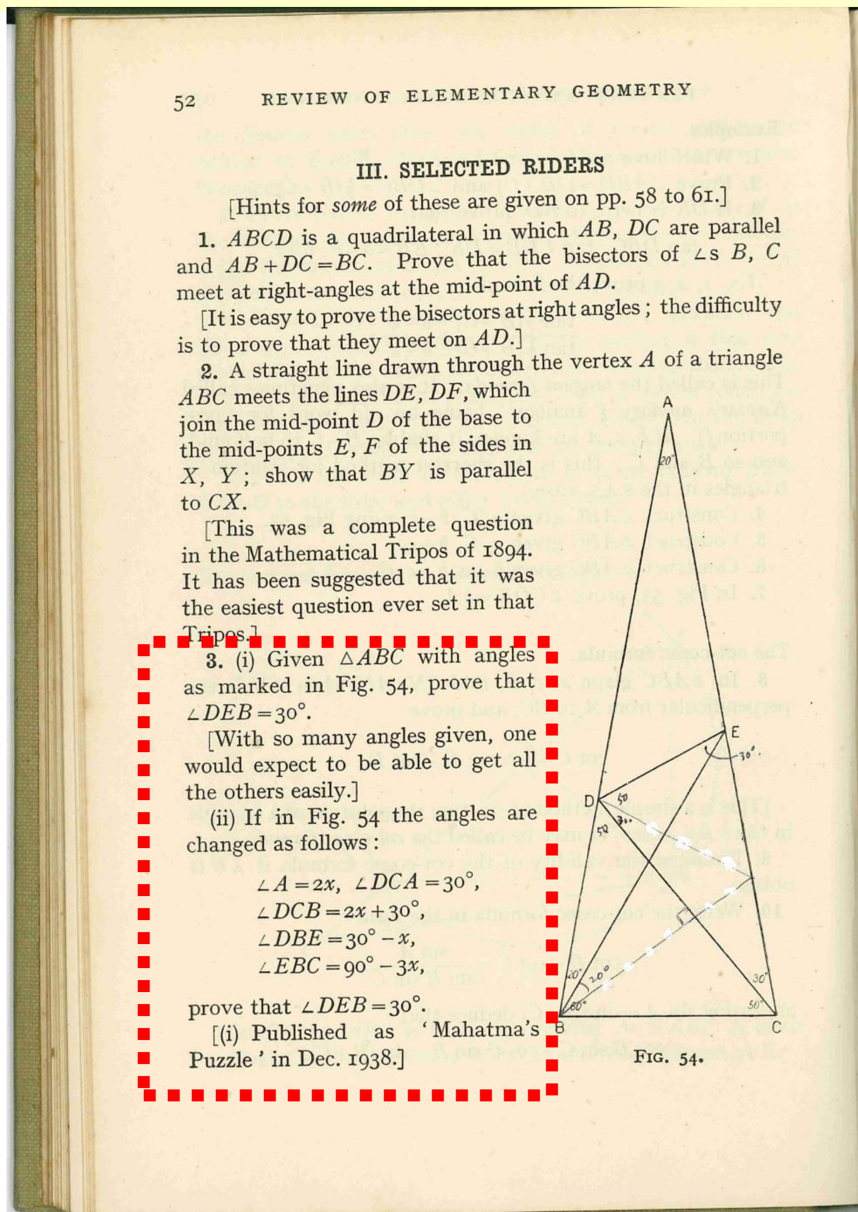
[Consider the orthogonal system of circles and use Ex. 472.]

C. Godfrey, A.W. Siddons, *Modern Geometry*, Cambridge University Press, 1954; first published in 1908.



" ..... Further, in many school certificate courses there is increasing emphasis on the practical aspect of geometry and in consequence much less time is now spent on formal and theoretical work than formerly. ..... "

C.O. Tuckey, F.J. Swan, *Geometry for Sixth Forms*, Longmans, Green & Co Ltd, 1948.



52 REVIEW OF ELEMENTARY GEOMETRY

III. SELECTED RIDERS

[Hints for *some* of these are given on pp. 58 to 61.]

1. $ABCD$ is a quadrilateral in which AB, DC are parallel and $AB+DC=BC$. Prove that the bisectors of $\angle$s B, C meet at right-angles at the mid-point of AD.

[It is easy to prove the bisectors at right angles; the difficulty is to prove that they meet on AD.]

2. A straight line drawn through the vertex A of a triangle ABC meets the lines DE, DF, which join the mid-point D of the base to the mid-points E, F of the sides in X, Y; show that BY is parallel to CX.

[This was a complete question in the Mathematical Tripos of 1894. It has been suggested that it was the easiest question ever set in that Tripos.]

3. (i) Given $\triangle ABC$ with angles as marked in Fig. 54, prove that $\angle DEB = 30°$.

[With so many angles given, one would expect to be able to get all the others easily.]

(ii) If in Fig. 54 the angles are changed as follows:

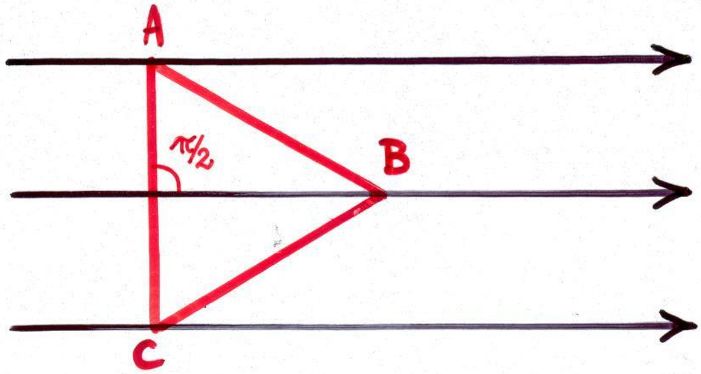
$$\angle A = 2x,\quad \angle DCA = 30°,$$
$$\angle DCB = 2x + 30°,$$
$$\angle DBE = 30° - x,$$
$$\angle EBC = 90° - 3x,$$

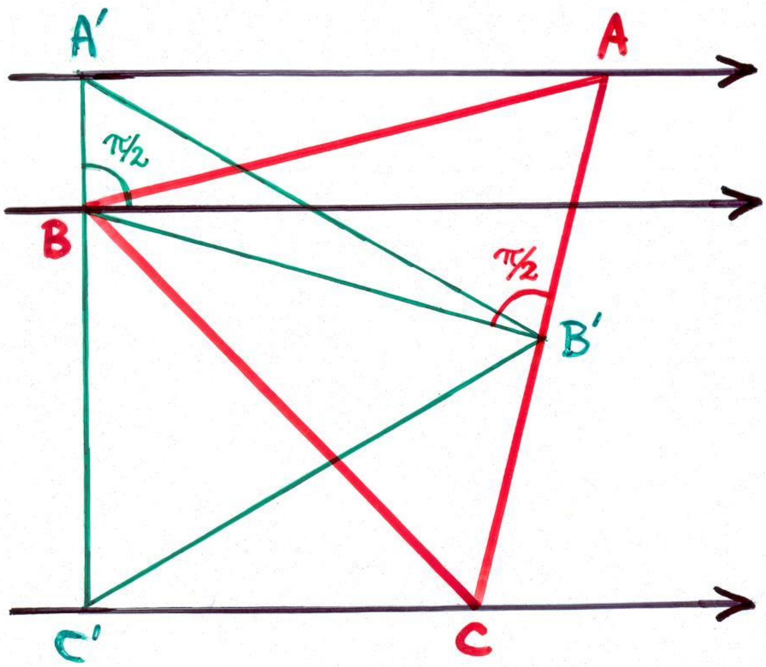
prove that $\angle DEB = 30°$.

[(i) Published as 'Mahatma's Puzzle' in Dec. 1938.]

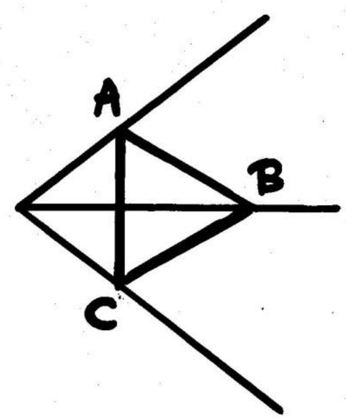
FIG. 54.

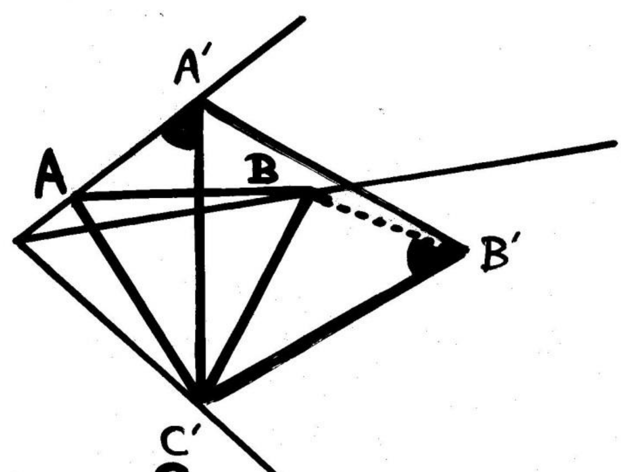
Given three parallel lines, construct an equilateral triangle with a vertex on each line. [Hint: What happens if the lines are equally spaced?]



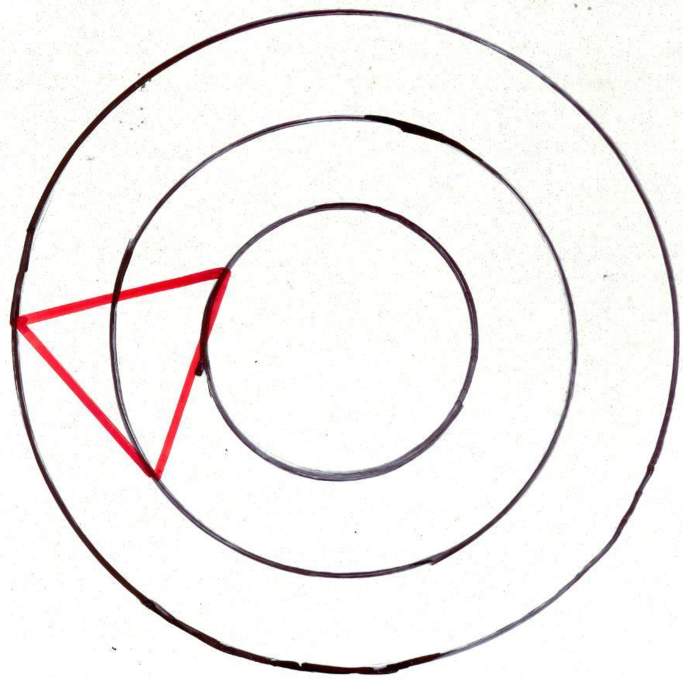


Given three straight lines intersecting at a point, construct an equilateral triangle with a vertex on each line.
[Hint: What happens if the lines are symmetrically placed?]





Given three concentric circles, construct an equilateral triangle with a vertex on each circle.



Q: Is the construction always possible?

A.B. Mayne, *The Essentials of School Algebra* (1938)

CHAPTER XVIII

GRAPHS (*Continued*)

113. Graphical solution of equations. In Chapter XIV, it was shown that by drawing the graph of $y=6x^2-7x-11$, it was possible to obtain approximate values of the roots of any equation of the type $6x^2-7x-11=a$, a being a constant. This is a particular case of a more general theorem, which we now proceed to discuss.

If we have a pair of simultaneous equations in x and y, and if the graphs corresponding to the equations are drawn with the same axes and with the same scales, then, at the points of intersection of the graphs,

1. **The coordinates are roots of the simultaneous equations ;**
2. **The x-coordinates are roots of the equation in x obtained by eliminating y from the two equations ;**
3. **The y-coordinates are roots of the equation in y obtained by eliminating x from the two equations.**

114. We shall prove these statements for a particular pair of equations, but it is clear that the method is quite general, provided that the eliminations can be performed.

Let us consider the equations $y=3x^2-6x+3$, $2x=3y-5$. The graphs corresponding to the equations are drawn in Fig. 17 on p. 223.

The graphs meet at P and Q, and PN, QM are the perpendiculars drawn from P and Q respectively to the axis Ox.

P lies on the curve, $\therefore\ NP=3\,.\,ON^2-6\,.\,ON+3.$ ....(i)

P lies on the st. line, $\therefore\ 2\,.\,ON=3\,.\,NP-5.$ ....(ii)

Thus, $x=ON$, $y=NP$ satisfy both the equations $y=3x^2-6x+3$ and $2x=3y-5$. It may similarly be shown that $x=OM$, $y=MQ$ satisfy these equations. This is the first result given above.

Also from (ii), $$NP=\frac{2\,.\,ON+5}{3}.$$

Substituting this value of NP in (i), we have
$$\frac{2\,.\,ON+5}{3}=3\,.\,ON^2-6\,.\,ON+3,$$
i.e. $x=ON$ satisfies the equation
$$\frac{2x+5}{3}=3x^2-6x+3. \quad \text{......................(iii)}$$
It may similarly be shown that $x=OM$ satisfies (iii).

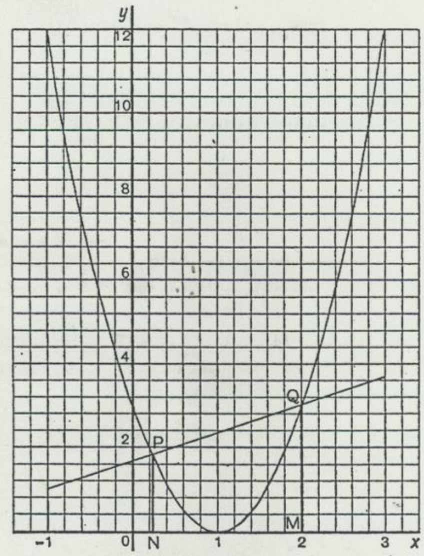


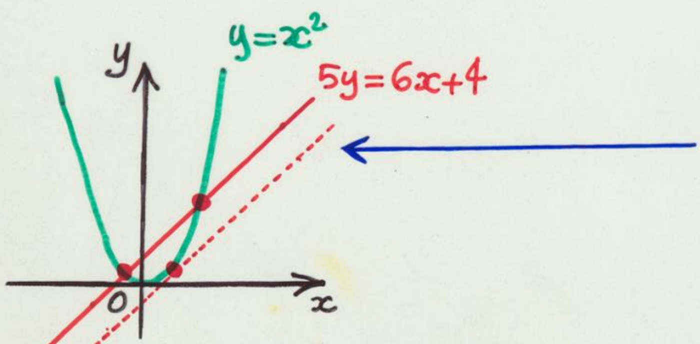
FIG. 17.

But (iii) is the equation obtained by eliminating y from the given equations. This is the second result given above.

Again, from (ii) $$ON=\frac{3\,.\,NP-5}{2}.$$

Substituting this value of ON in (i), we have
$$NP=3\left(\frac{3\,.\,NP-5}{2}\right)^2-6\left(\frac{3\,.\,NP-5}{2}\right)+3,$$

24. Draw the graphs of $y=x^2$ and $5y=6x+4$ on the same diagram for values of x from -2 to 3. From the graphs solve $5x^2=6x+4$. Also find out roughly from the graphs, by drawing the appropriate parallel line, for what value of a the equation $5x^2=6x+a$ will have equal roots.



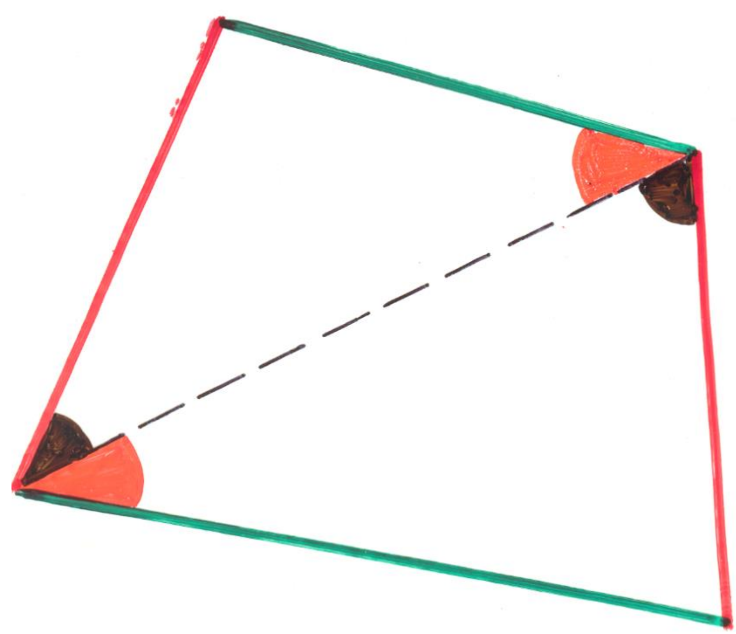
$5y=6x+a$

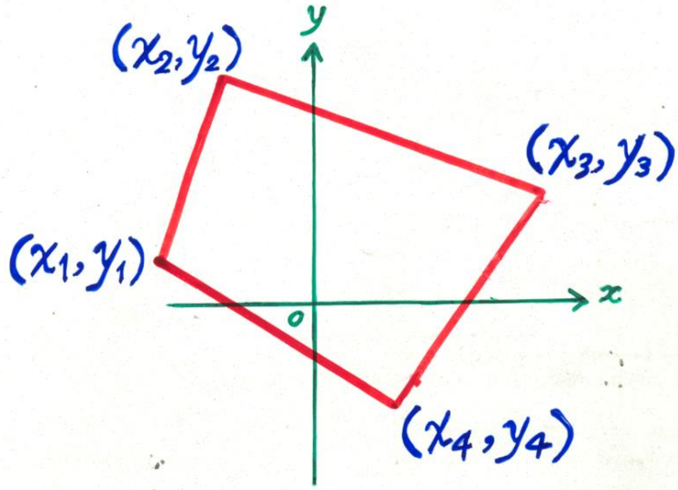
$y=\frac{6}{5}x+\frac{a}{5}$

Read off from the graph $\frac{a}{5}=$?

$a=5\times$?

If each pair of opposite sides of a quadrilateral are equal, is the quadrilateral a parallelogram?

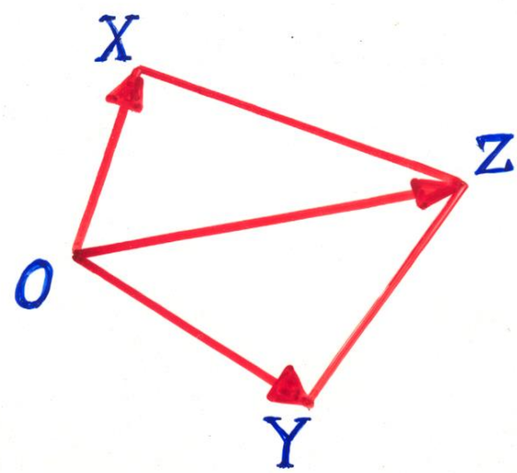




Given: $(x_2-x_1)^2+(y_2-y_1)^2=(x_4-x_3)^2+(y_4-y_3)^2$

$(x_4-x_1)^2+(y_4-y_1)^2=(x_3-x_2)^2+(y_3-y_2)^2$

To prove: $(y_2-y_1)(x_4-x_3)=(y_4-y_3)(x_2-x_1)$



Given: $|X| = |Z-Y|$ and $|Y|=|Z-X|$

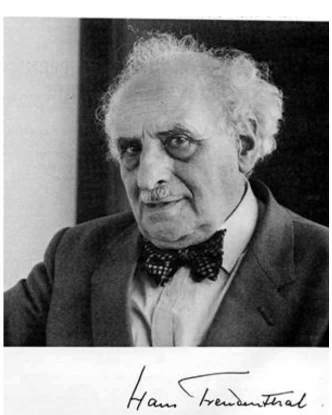
To prove: $X+Y=Z$

幾何的世界，既真實也抽象。如何能夠在數學課堂構建一個幾何的世界，搭建起抽象世界和真實世界之間的橋樑，以彌合虛擬（抽象的、理論的）世界和真實（具體的）世界之間的差距？透過這樣做，讓初學者感到自在，而不是脫離了生活。

一些有關幾何教學長期富爭議的問題：

- 經驗知識（“物理”幾何）和理論知識（“純”幾何）
- 啟發式解釋和正式證明
- 直覺和演繹推理
- 空間的理解和計算技能

Geometry is grasping space … grasping that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it. …Geometry is one of the best opportunities that exists to learn how to mathematize reality. It is an opportunity to make discoveries …teaching geometry is an unparalleled struggle between ideal and realization.

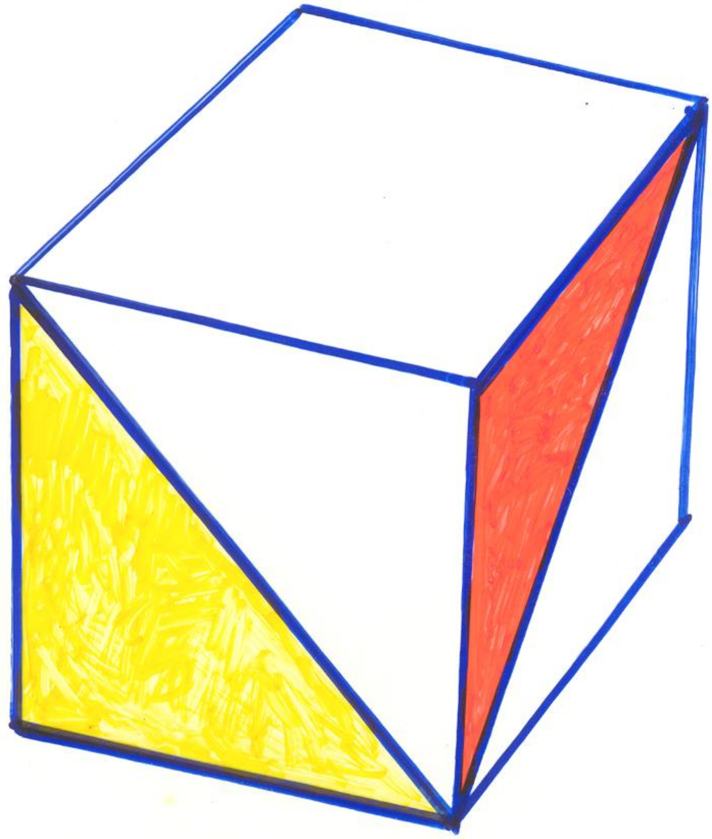


H. Freudenthal, *Mathematics as an Educational Task* (1973), Chapter XVI: The Case of Geometry.

"幾何眼(geometrical eye)" ＝ "即使脫離了圖形也能看出其中幾何性質的能力"

C. Godfrey, The Board of Education circular on the teaching of geometry, *Math. Gazette*, 5 (1910), 195-200.

C. Godfrey, A.W. Siddons, *Elementary Geometry, Practical and Theoretical* (1903).



N. Rouche, Reaction to papers on geometry, in *One Hundred Years of L'Enseignement Mathématique: Moments of Mathematics Education in the Twentieth Century*, ed. D. Coray et al, 2003, p.156.

這兩個三角形的面積是否相等？

幾何在不同的領域都會出現

★ 中學代數 (運用圖形解題，座標幾何)

★ 線性代數 ([Euclidean]內積空間) — 這是基於 Euclid – Hilbert – Dedekind 的幾何公理系統）

★ 微積分 (曲線和曲面)

★ 物理 (包括 Gauss、Riemann、Poincaré 及Einstein 的工作)

★ 抽象代數、數論、組合數學，(包括有限幾何、代數結構、Diophantus方程、 …)

1869 **James Joseph Syvester (1814-1897)**

— a plea for the mathematician, *Nature*, 1 (1869-70).

1868 **James Maurice Wilson (1836-1931)**

— *Elementary Geometry*.

1868 **August De Morgan (1806-1871)**

— Review on Wilson's textbook.

Charles Lutwidge Dodgson (1832-1898)

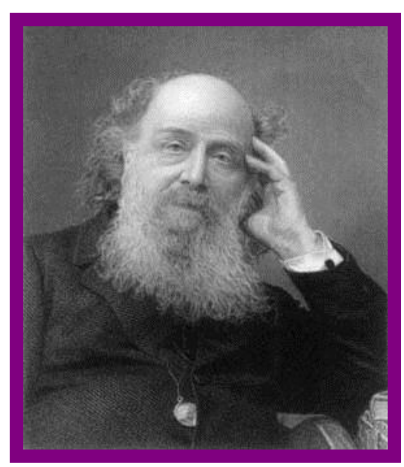
— *Euclid and his Modern Rivals*.

Isaac Todhunter (1820-1884)

— *Euclid for the Use of Schools and Colleges: Comprising the First Six Books and Portions of the Eleventh and Twelfth Books* **(1862).**

"The early study of Euclid made me a hater of Geometry, which I hope may plead my excuse if I have shocked the opinions of any in this room (...) by the tone in which I have previously alluded to it as a schoolbook;"

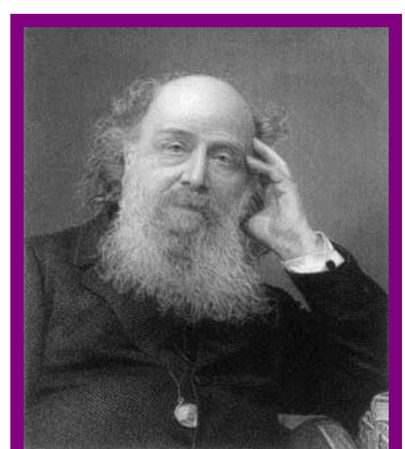
The Collected Mathematical Papers of James Joseph Sylvester, **Volume II, edited by H.F. Baker (four volumes, 1904-1910).**



James Joseph Sylvester (1814-1897)

"and yet, in spite of this repugnance, which had become a second nature in me, whenever I went far enough into any mathematical question, I found I touched, at last, a geometrical bottom."

The Collected Mathematical Papers of James Joseph Sylvester, Volume II, edited by H.F. Baker (four volumes, 1904-1910).



James Joseph Sylvester (1814-1897)

1870 Meeting of 36 headmasters of public schools

1871 Special Committee of the British Association for the Advancement of Science

Arthur Cayley (1821-1895)
T. Archor Hirst (1830-1892)
William Kingdon Clifford (1845-1879)
George Salmon (1819-1904)
Henry John Stanley Smith (1826-1883)
James Joseph Sylvester (1814-1897)
James Maurice Wilson (1836-1931)

1871 **AIGT (Association for the Improvement of Geometrical Teaching)**

[became Mathematical Association in 1897, with official journal *Mathematical Gazette* started in 1894]

- **syllabus of plane geometry in 1875.**
- **AIGT Reports issued through 1893.**

1901 **John Perry (1850-1920)**

– **The teaching of mathematics, *Educational Review*, 23 (1902),158-181.**

1903 **Sequence of Euclid no longer enforced in examinations in Oxford University and Cambridge University.**

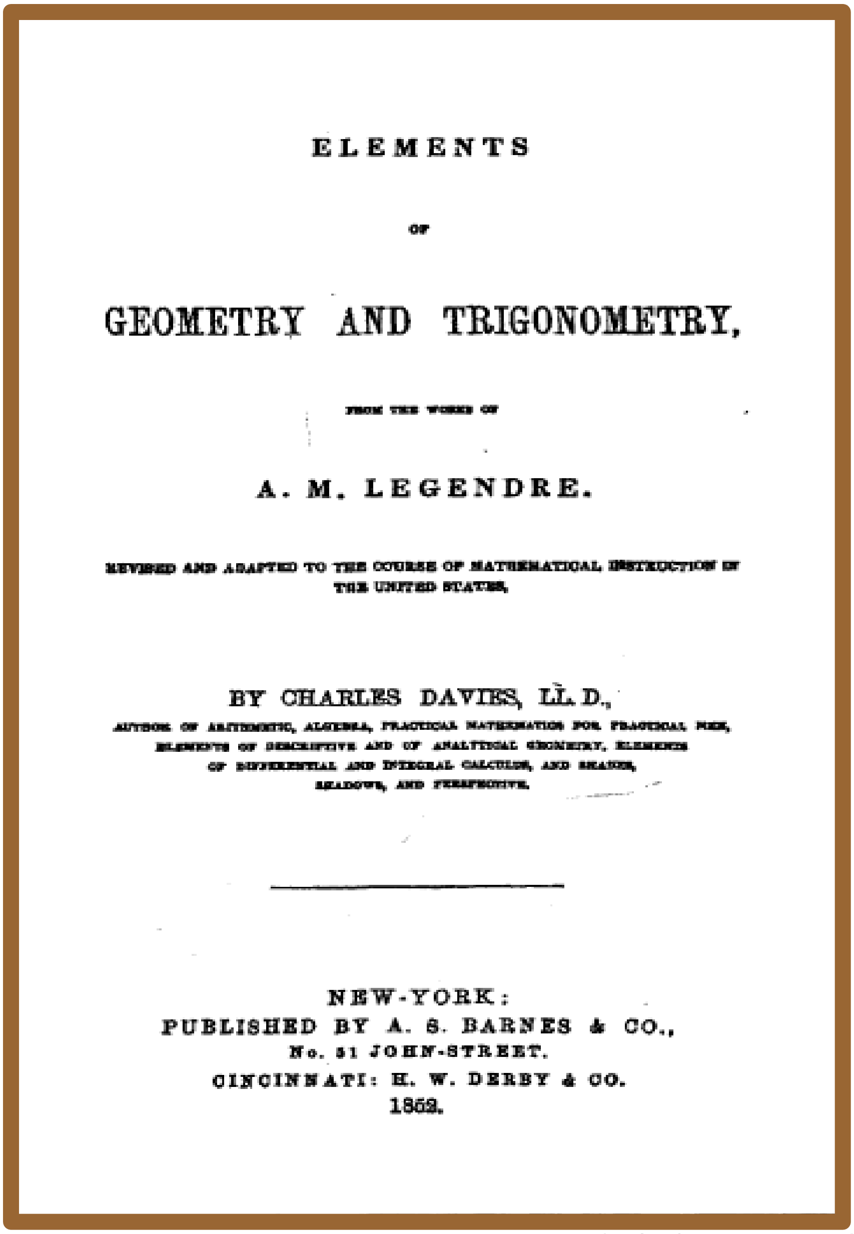
“A bas Euclide (Down with Euclid)!”

[famous slogan of J.A. Dieudonné at the Royaumont Seminar of OECE (now OECD) in November of 1959]

- **R. Thom, Les mathématiques modernes: Une erreur pédagogique et philosophique*? L’Age de la Science*, 3 (1970), 225-236; A*merican Scientists*, 59 (1971), 695-699.**
- **J.A. Dieudonné, Should we teach “modern” mathematics? *American Scientists*, 61 (1973), 16-19.**

中譯文見於「新數學課程的爭議」,《抖擻》,
13 (1976), 29-39.

- **G. Howson, Geometry: 1950-70, in *One Hundred Years of L'Enseignement Mathématique*, edited by D. Coray *et al*, 2003, 115-131.**


ELEMENTS

OF

GEOMETRY AND TRIGONOMETRY,

FROM THE WORKS OF

A. M. LEGENDRE.

REVISED AND ADAPTED TO THE COURSE OF MATHEMATICAL INSTRUCTION IN THE UNITED STATES,

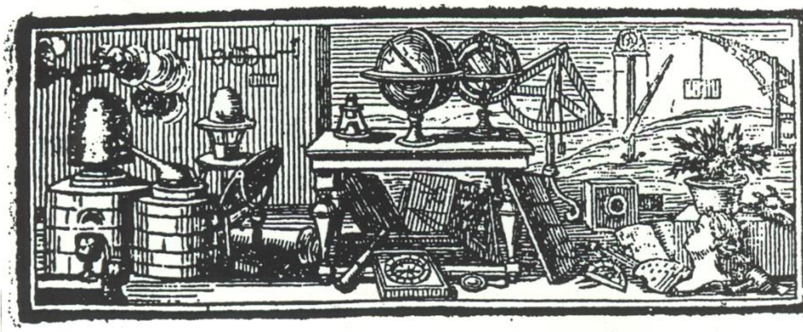
BY CHARLES DAVIES, LL. D.,

AUTHOR OF ARITHMETIC, ALGEBRA, PRACTICAL MATHEMATICS FOR PRACTICAL MEN, ELEMENTS OF DESCRIPTIVE AND OF ANALYTICAL GEOMETRY, ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS, AND SHADES, SHADOWS, AND PERSPECTIVE.

NEW-YORK:
PUBLISHED BY A. S. BARNES & CO.,
No. 51 JOHN-STREET.
CINCINNATI: H. W. DERBY & CO.
1852.

Adrien-Marie Legendre, *Eléments de géométrie* (1794; many editions; English translation by Charles Davies in 1852)

"Davies' Legendre" means geometry in the United States in the second half of the nineteenth century!



ÉLÉMENS
DE
GÉOMÉTRIE.

PREMIERE PARTIE.

Des moyens qu'il étoit le plus naturel d'employer pour parvenir à la meſure des Terreins.

CE qu'il ſemble qu'on a dû meſurer d'abord, ce ſont les longueurs & les diſtances.

I.

POUR meſurer une longueur quelconque, l'expédient que fournit une

A

Alexis Claude Clairaut
(1713-1765)

A.C. Clairaut, Elements de géométrie (1741; 1753)

「雖然幾何本身是一個抽象的學科，但是必須承認，導致初學者沮喪的困難，大多源於按照初等課本的教授方式。它們一開始總是搬出大量的定義，公設，公理和一些預備原則，這些似乎毫無用途的敘述，只會使讀者感覺枯燥。接著而來的定理或者沒有引起讀者的興趣，或者是難於理解的。因此，最後初學者身心疲憊而放棄，之前他們完全不明白課本要教他們什麼。」

Alexis Claude Clairaut (1713-1765)

***A.C. Clairaut, Elements de géométrie* (1741; 1753)**

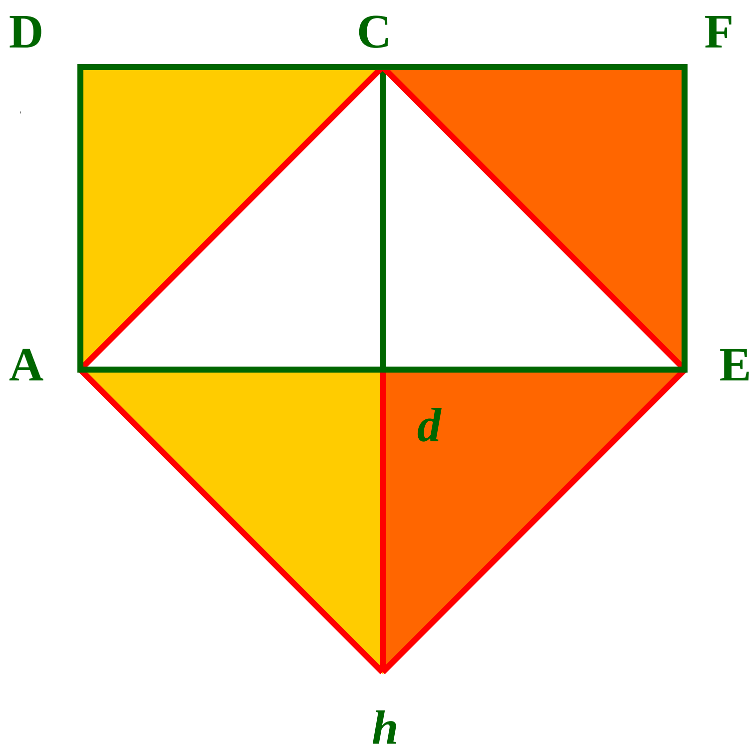
「...但我希望還有另外一個重要的用途，它使讀者養成探索和發現的習慣，因為我小心避免以定理的形式來陳述一個命題；也就是說，我避免只證明命題的真實性卻沒有呈現它被發現的方式。」

Alexis Claude Clairaut (1713-1765)

A.C. Clairaut, Elements de géométrie (1741; 1753)

A.C. Clairaut, *Eléments de géométrie* (1741; 1753)

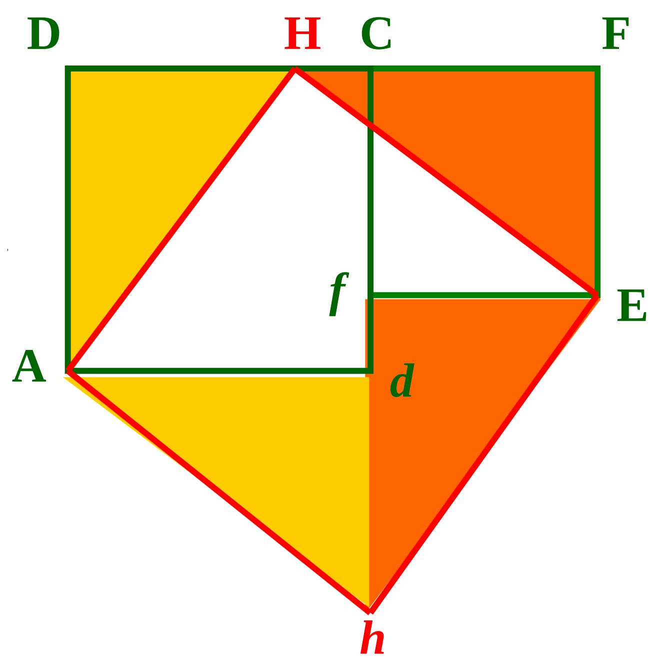
XVI To make a square equal in area to two equal (smaller) squares.



ACEh is a square

ACEh = ADCd + CFEd

XVII To make a square equal in area to two other taken together.

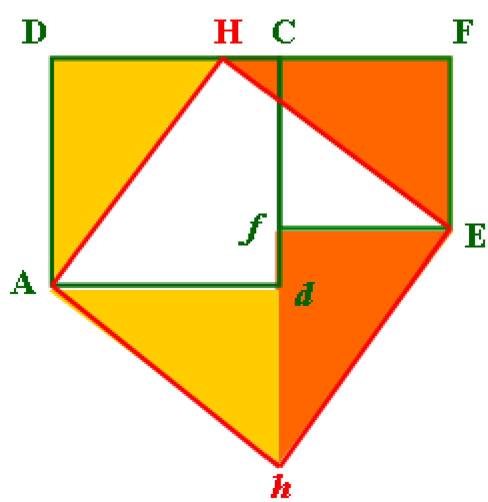


Following the trend of thought in XVI we try to find a point H on DF such that
(i) when ADH is turned around A and when EFH is turned around E, they join at a point *h*.
(ii) AH, HE, E*h*, *h*A are equal and perpendicular.

Take H on DF such that DH = CF = EF.

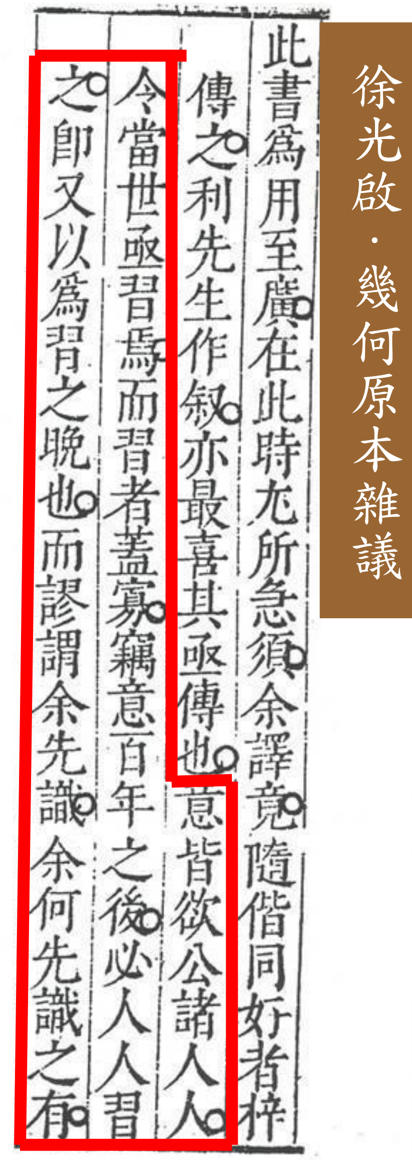
XVIII **The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two other sides.**

(Pythagoras Theorem)



「此書為用至廣，在此時尤所急須。…利先生作敘，亦是最喜其亟傳也。意皆欲公諸人人，令當世亟習焉，而習者蓋寡。竊意百年之後必人人習之，即又以為習之晚也，而謬謂余先識，余何先識之有。」

徐光啟・《幾何原本雜議》(1607)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傳之。利先生作叙。亦最喜其亟傳也。意皆欲公諸人人。
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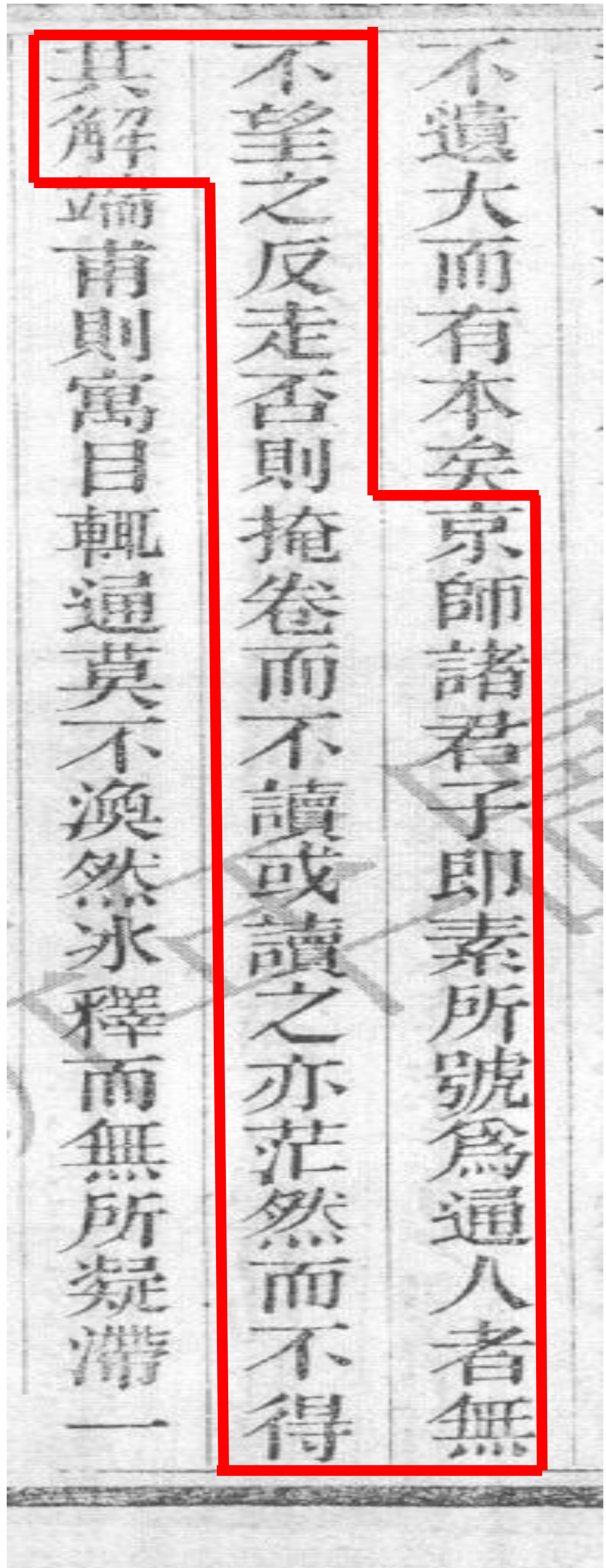
之。即又以爲習之晚也。而謬謂余先識。余何先識之有。

徐光啟・幾何原本雜議

「京師諸君子即素所號為通人者，無不望之反走，否則掩卷而不讀，或讀之亦茫然而不得其解。」

李子金序，《數學鑰》(杜知耕，1681)


不遺大而有本矣京師諸君子即素所號爲通人者無
不望之反走否則掩卷而不讀或讀之亦茫然而不得
其解端甫則寓目輒通莫不渙然冰釋而無所疑滯一

「自明之末葉，利瑪竇等輸入當時所謂西學者於中國，而學問研究方法上，生一種外來的變化。其初惟治天算者宗之，後則漸應用於他學。」

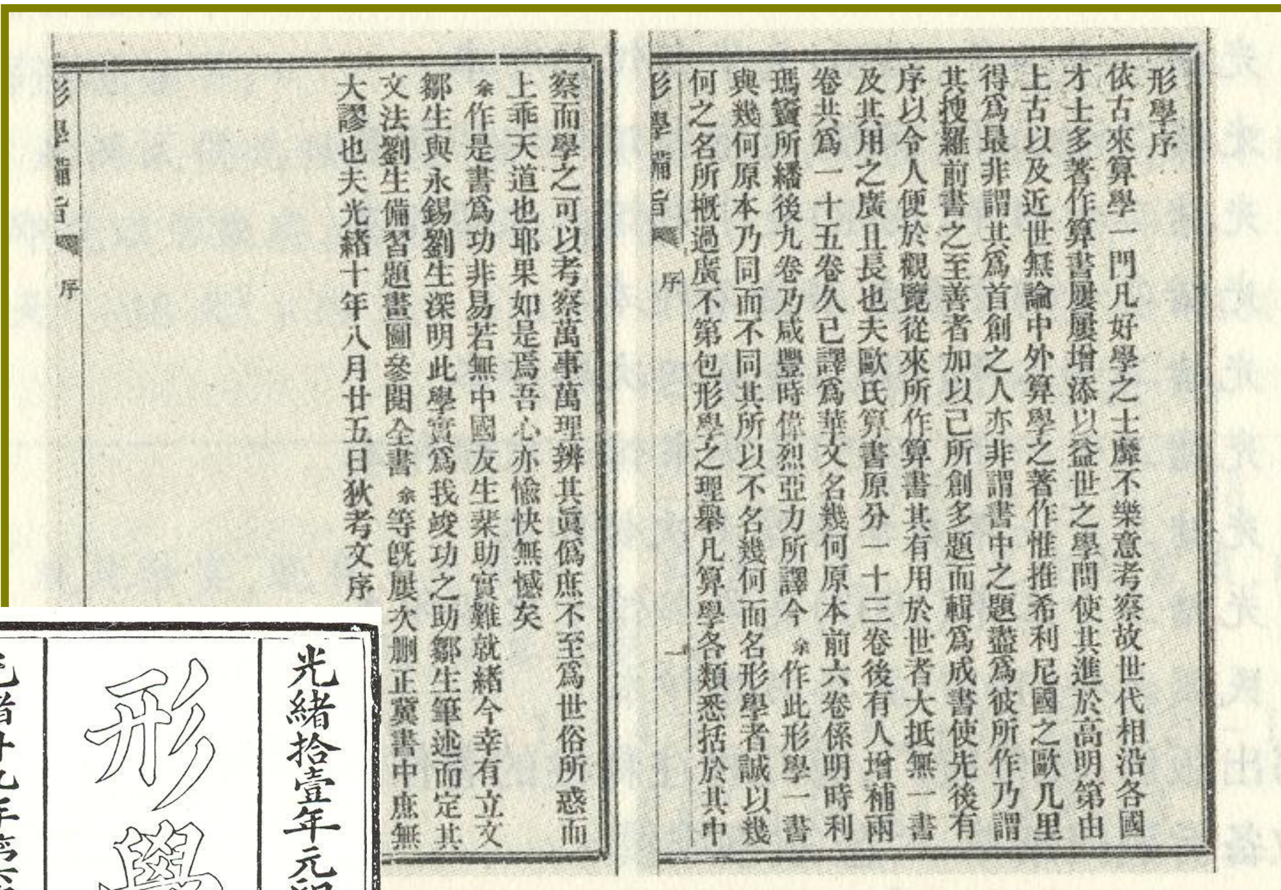
梁啟超，《清代學術概論》
(原刊載於《改造雜誌》
1920, 1921)

「《幾何原本》，徐交定僅譯前六卷，至李壬叔乃續成之。然第十卷之理甚深，非初學者所能解。即西人學校通習者，亦僅在前六卷。」

「故偉烈亞力謂西人欲求此書善本，當反索之中國矣。學者初但觀徐譯，久之此學日深，神明其法，自能讀全書也。（《數理精蘊》本較簡，然究以讀原書為佳。）」

「《形學備旨》序，謂有許多要題，乃近世新得，不在《幾何原本》之內者，西國每譯幾何，必將要題增補於各卷之後，今李譯皆無之云云。然則讀幾何者，不得不兼讀此書矣。」

梁啟超，西學書目表序例
《時務報》第八冊 (1896)



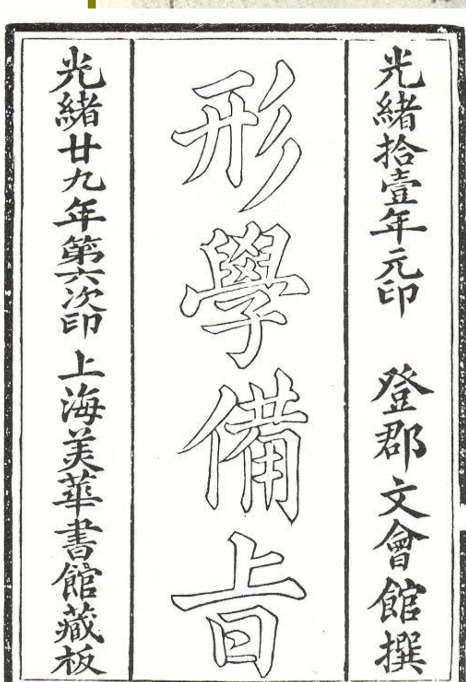
形學序

依古來算學一門凡好學之士靡不樂意考察故世代相沿各國
才士多著作算書屢屢增添以益世之學問使其進於高明第由
上古以及近世無論中外算學之著作惟推希利尼國之歐几里
得爲最非謂其爲首創之人亦非謂書中之題盡爲彼所作乃謂
其搜羅前書之至善者加以己所創多題而輯爲成書使先後有
序以令人便於觀覽從來所作算書其有用於世者大抵無一書
及其用之廣且長也夫歐氏算書原分一十三卷後有人增補兩
卷共爲一十五卷久已譯爲華文名幾何原本前六卷係明時利
瑪竇所繙後九卷乃咸豐時偉烈亞力所譯今余作此形學一書
與幾何原本乃同而不同其所以不名幾何而名形學者誠以幾
何之名所概過廣不第包形學之理舉凡算學各類悉括於其中

形學備旨 序 一

察而學之可以考察萬事萬理辨其眞僞庶不至爲世俗所惑而
上乖天道也耶果如是爲吾心亦愉快無憾矣
余作是書爲功非易若無中國友生襄助實難就緒今幸有立文
鄒生與永錫劉生深明此學實爲我竣功之助鄒生筆述而定其
文法劉生備習題畫圖參閱全書余等既屢次删正冀書中庶無
大謬也夫光緒十年八月廿五日狄考文序

形學備旨 序



光緒拾壹年元印

形學備旨

登郡文會館撰

光緒廿九年第六次印 上海美華書館藏板

***The Complete Meaning of the Science of Figures* (形學備旨), compiled by Calvin Wilson Mateer (狄考文) and Zou Li-wen (鄒立文) [believed to be a selected translation of a book by Elias Loomis], 1885**

夫歐幾里得之書，條理統系，精密絕倫，非僅論數論象之書，實為希臘民族精神之所表現。此滿文譯本及數理精蘊本皆經刪改，意在取便實施，而不知轉以是失其精意。

陳寅恪，幾何原本滿文譯本跋［原載歷史語言研究所集刊第二本第三分(1931)］

陳寅恪 (1890-1969)

徐光啟 ● 幾何原本雜議

下學工夫，有理有事。此書爲益，能令學理者祛其浮氣，練其精心；學事者資其定法，發其巧思，故舉世無一人不當學。聞西國古有大學，師門生常數百千人，來學者先問能通此書，乃聽入。何故？欲其心思細密而已。其門下所出名士極多。

能精此書者，無一事不可精；好學此書者，無一事不可學。

凡他事、能作者能言之，不能作者亦能言之；獨此書爲用，能言者即能作者，若不能作，自是不能言。何故？言時一毫未了，向後不能措一語，何由得妄言之。以故精心此學，不無知言之助。

此書為益，能令學理者**祛其浮氣**，**練其精心**，學事者**資其定法**，**發其巧思**，故**舉世無一人不當學**。

凡人學問，有解得一半者，有解得十九或十一者，獨幾何之學，通即全通，蔽即全蔽，更無高下分數可論。

人具上資而意理疏莽，即上資無用；人具中材而心思縝密，即中材有用，能通幾何之學，縝密甚矣！故率天下之人而歸於實用者，是或其所由之道也。

此書有四不必㊂：不必疑，不必揣，不必試，不必改。有四不可得：欲脫之不可得，欲駁之不可得，欲減之不可得，欲前後更置之不可得。有三至、三能：似至晦實至明，故能以其明明他物之至晦；似至繁實至簡，故能以其簡簡他物之至繁；似至難實至易，故能以其易易他物之至難。易生于簡，簡生于明，綜其妙在明而已。

「四不必，四不可得」。此評語是否說過了頭?

此書爲用至廣，在此時尤所急須，余譯竟，隨偕同好者梓傳之。利先生作敍，亦最喜其亟傳也，意皆欲公諸人人，令當世亟習焉。而習者蓋寡，竊意百年之後必人人習之，即又以爲習之晚也。而謬謂余先識，余何先識之有？

有初覽此書者，疑奧深難通，仍謂余當顯其文句。余對之：度數之理，本無隱奧，至于文句，則爾日推敲再四，顯明極矣。倘未及留意，望之似奧深焉，譬行重山中，四望無路，及行到彼，蹊徑歷然。請假旬日之功，一究其旨，即知諸篇自首迄尾，悉皆顯明文句㊂。

幾何之學，深有益於致知。明此，知向所揣摩造作，而自詭爲工巧者皆非也。一也。明此，知吾所已知不若吾所未知之多，而不可算計也。二也。明此，知向所想像之理，多虛浮而不可挼也。三也。明此，知向所立言之可得而遷徙移易也㊃。

此書有五不可學：躁心人不可學，麤心人不可學，滿心人不可學，妬心人不可學，傲心人不可學。故學此者不止增才，亦德基也。

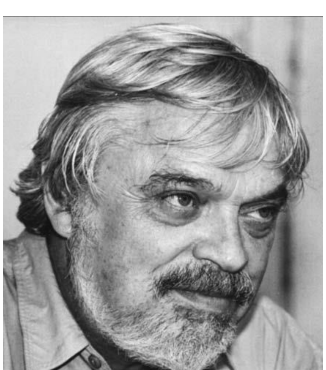
3

昔人云：「鴛鴦繡出從君看，不把金針度與人」，吾輩言幾何之學，政與此異。因反其語曰：「金針度去從君用，未把鴛鴦繡與人」，若此書者，又非止金針度與而已，直是教人開丱冶鐵，抽線造針，又是教人植桑飼蠶，湅絲染縷。有能此者，其繡出鴛鴦，直是等閑細事。然則何故不與繡出鴛鴦？曰：能造金針者能繡鴛鴦，方便得鴛鴦者誰肯造金針？又恐不解造金針者，菟絲棘刺，聊且作鴛鴦也！其要欲使人人真能自繡鴛鴦而已。

鴛鴦繡出從君看　不把金針度與人

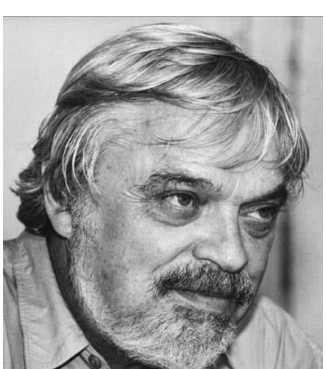
金針度去從君用　未把鴛鴦繡與人

“Geometry is a phenomenon of the human culture. … Geometry, as well as mathematics in general, helps in moral and ethical education of children. … Geometry develops mathematical intuition, introduces a person to independent mathematical creativity. … Geometry is a point of minimum for the distance between school mathematics and the mathematics of high level.”



Igor Fedorovich Sharygin (沙雷金) (1937-2004)

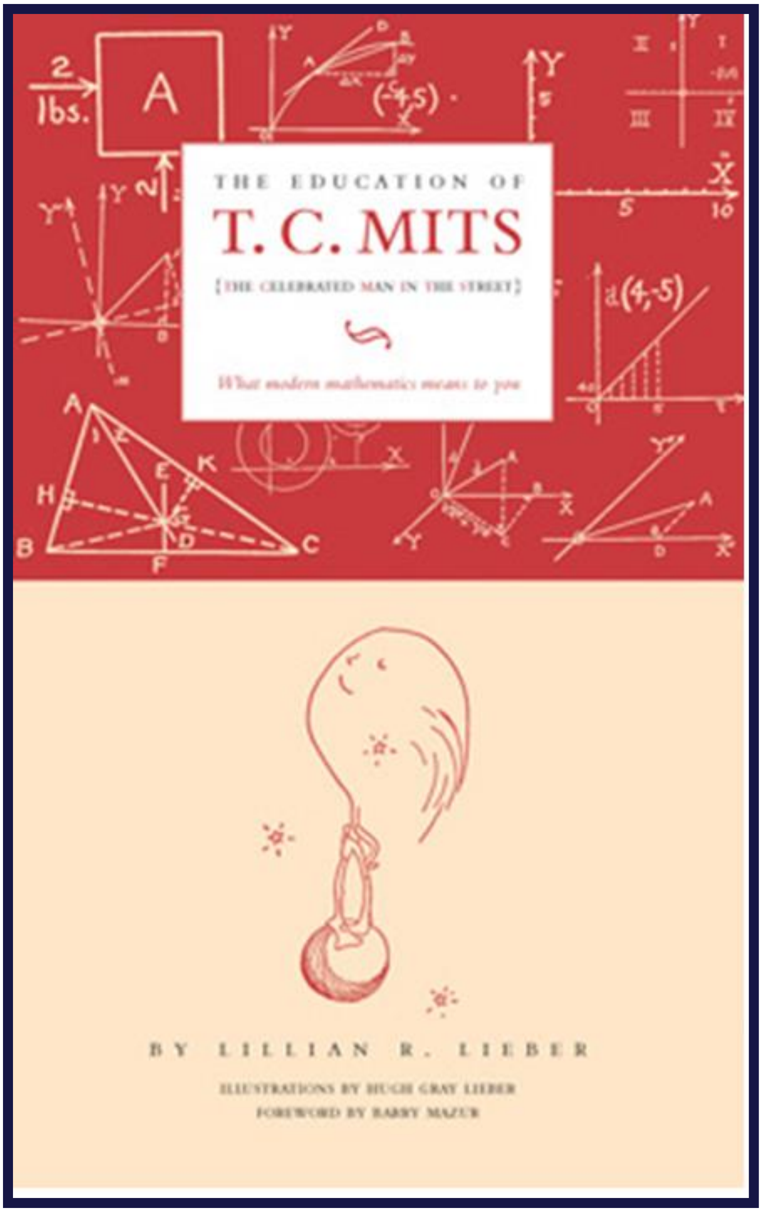
“Learning mathematics builds up our virtues, sharpens our sense of justice and our dignity, strengthens our innate honesty and our principles. The life of mathematical society is based on the idea of proof, one of the most highly moral ideas in the world .”



Igor Fedorovich Sharygin (沙雷金) (1937-2004)

"Rigour is to the mathematician what morality is to man.
(嚴謹之於數學家，猶如道德之於一般人。)"

André Weil
(1906-1998)

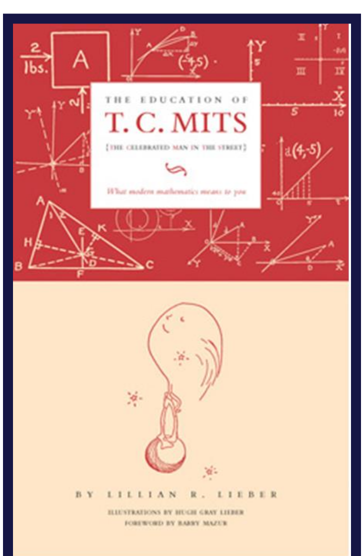


Lillian R. Lieber, *The Education of T.C. Mits: What Modern Mathematics Means to You*, originally published in 1942; republished in 2007.

[T.C. Mits = The Celebrated Man In The Street]

Lillian Rosanoff Lieber, (1886-1986)

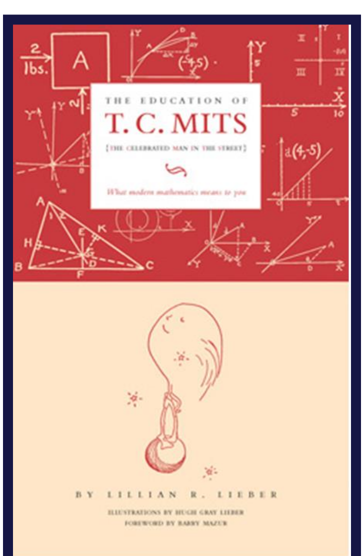
And so you see how
Mathematics can throw light
on various subjects
which many people discuss
glibly and carelessly
since they have never been trained
to examine ideas
With that METICULOUS CARE
With which a mathematician works.



Lillian R. Lieber, *The Education of T.C. Mits: What Modern Mathematics Means to You*, originally published in 1942; republished in 2007.

THERE is a model for straight thinking
Which we all MUST try to imitate.
This is not the
noisy argumentation of
the pseudo-thinkers.
Rather it is
quiet,
honest,
careful,
COMPETENT.

The Moral: Do not be NAÏVE —
Use the methods of
Mathematics.

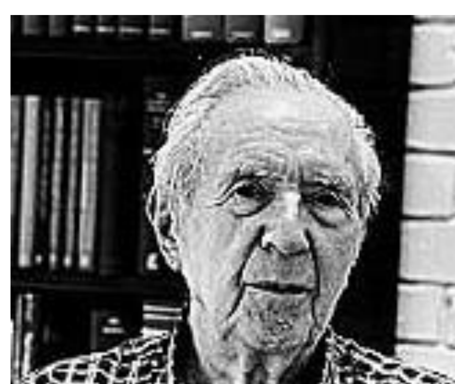


Lillian R. Lieber, *The Education of T.C. Mits: What Modern Mathematics Means to You*, originally published in 1942; republished in 2007.

Teaching is not a lost art, but the regard for it is a lost tradition.

(教學不是逝去了的藝術，然而對它的尊重卻是逝去了的傳統。)

Jacques Barzun,
***Teacher in America*, 1945, p.12**
[quoted in Newsweek, December 5, 1955]



Jacques Barzun
(1907-2012)

❖ **M.K. Siu, When “Mr. Ou (Euclid)” came to China… , to appear in the P*roceedings of the 6th International Congress of Chinese Mathematicians, July 2013, Taipei.***
[Chinese translation in : 當「歐先生」來到中國… , 數學傳播, 38(4) (2014), 24-41.]

❖ **M.K. Siu, The world of geometry in the classroom: Virtual or real? *Proceedings of 5th International Colloquium on the Didactics of Mathematics*, Vol. II, edited by M. Kourkoulos, C. Tzanakis, University of Crete (2009), 93-112.**
[Chinese translation in: 課堂中的幾何世界: 虛擬還是真實? 中學數學, 346 (2009), 1-4, 46; 348 (2009), 1- 6.]

❖ **P. Engelfriet, M.K. Siu, Xu Guangqi's attempts to integrate Western and Chinese mathematics, in *Statecraft And Intellectual Renewal In Late Ming China: The Cross-Cultural Synthesis of Xu Guangqi (1562-1633)*, edited by C. Jami, P. Engelfriet, G. Blue, Brill, Leiden, 2001, 279-310.**

謝謝香港數理教育學會的邀請，讓我有此機會與大家談談數學。

謝謝梁子傑老師應允作回應嘉賓，與大家分享他的高明識見。

柯志明先生協助製作 *GeoGebra* 顯示，以輔助講解，香港大學數學系呂美美女士協助製作圖片，為講座添色，謹此一併致謝。